Final Exam - Analyse II - Wiskunde

Wednesday, June 20, 2018, 14.00-17.00

- Write your name and student ID in a clearly readable manner on each page.
- Every answer has to be motivated by a computation, explanation of reasoning or reference to the theory.
- Calculators may be used only if non-graphical.

This exam has *four* questions.

Question 1

Consider the scalar function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = (4x^2 + y^2 + 1)e^{(-x^2-y^2)/4}$$

and the domain

$$\mathcal{D} = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{1}{2}x^2 + \frac{1}{3}y^2 \leq 1\right\} \subset \mathbb{R}^2 .$$

(a) Determine critical points of f on $\mathbb{R}^2$ and classify these (maximum/minimum/saddle point).

(b) Find candidates for maxima/minima of f on the boundary $\partial\mathcal{D}$.

(c) Find all global maxima/minima of f on $\mathcal{D}$.

Question 2

Consider the vector field

$$F(x, y, z) = \left(9y^2x, 4x^2y, \frac{1}{3}z^3 - 36z\right)$$

and the surface

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid 4x^2 + 9y^2 + z^2 = 36\} ,$$

oriented such that the normal vectors point inwards.

(a) Use the divergence theorem to compute the flux of F through $\mathcal{S}$

$$\iint_{\mathcal{S}} F \cdot dS = \iint_{\mathcal{S}} F \cdot \hat{N} dS .$$

(b) Consider the integral

$$\iint_{\widetilde{\mathcal{S}}} F \cdot dS$$

for an arbitrary smooth surface $\widetilde{\mathcal{S}} \subset \mathbb{R}^3$ that does not intersect itself and encloses a bounded domain. Determine the maximal value that this integral can attain and give the surface $\widetilde{\mathcal{S}}$ and corresponding enclosed domain for which this maximum is attained.

Please turn the page. There are more questions on the back!

Question 3

Consider the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y) = \left(5x + 3y, \frac{1}{5}y\right).$$

The curve $\mathcal{C}$ is the path from $(0, 5)$ to $(0, -5)$ along the half ellips $25x^2 + y^2 = 25$ with $x \leq 0$.

(a) Compute the vector line integral

$$\int_{\mathcal{C}} F(r) \cdot dr$$

through direct calculation.

(b) Compute the vector line integral in (a) again, but this time using the fact that $(5x, \frac{1}{5}y)$ is conservative.

(c) Compute the vector line integral in (a) again, but this time using Green's theorem.

Question 4

Consider the vector field

$$F(x, y, z) = \big(z(x + y), -z(x + y), z\big)$$

together with the surface

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : 2z = 2x^2 - xy + 2y^2 \text{ and } 0 \leq z \leq 8\},$$

oriented such that the normal vector points upwards.

(a) Compute the vector line integral

$$\oint_{\partial\mathcal{S}} F(r) \cdot dr$$

directly. *Hint: As usual, $\partial\mathcal{S}$ denotes the surface boundary of $\mathcal{S}$, positively oriented w.r.t. $\mathcal{S}$.*

(b) Compute the vector line integral in (a) again, but this time using Stokes' theorem.