Inleiding kansrekening

Teacher: L. Avena
Written examination: Friday 15 June 2018, 14:00–17:00.

- Write your name and student identification number **on each piece of paper** you hand in.
- All answers **must come with a full explanation**. Formulas alone are not enough. Formulate your answers clearly and carefully.
- The use of textbooks, lecture notes or handwritten notes **is not allowed**.
- To each question is associated a score written in boldface e.g. in **Exercise 1**, [**3**] points are assigned to question (a). *Total score:* 100. Pass: ≥ 55; no pass: ≤ 54.

Exercise 1

In a tennis tournament there are 2^n players, with $n \geq 2$. In each match of the tournament, a player either wins or it loses (drawn can not happen). All players are equally strong, namely, each participant has probability $1/2$ to win against any other opponent in a single match. In the first round of the tournament, the players are grouped in $2^n - 1$ pairs randomly chosen. The winners go on to the second round, while the losers are directly eliminated. The next rounds proceed analogously until only one player is left: the champion. In other words, in each next round, the winners of the previous one are paired at random. YOU and NADAL are playing this tournament.

(a) [**3**] What is the probability that YOU will be the champion?

(b) [**4**] What is the probability that YOU will face and win against NADAL at the i-th round?

(c) [**3**] What is the probability that YOU and NADAL will face each other during the tournament?

Exercise 2

A box contains 100 dice, 30 of which are defective. A defective die shows the face with number 2 with probability 1/9 and the face with number 5 with probability 2/9 (all other faces with probability 1/6). A die is taken uniformly at random from the box and rolled repeatedly.

(a) **[10]** If at the second throw the face with number 2 appears, what is the probability that this die is defective?

Exercise 3

Consider 5 biased coins such that the i-th coin shows a HEAD with probability $i/5$, for $i \in \{1, 2, 3, 4, 5\}$. Pick one of these coins uniformly at random and let N denote the number of HEADS appearing in 10 independent tosses of this coin.

(a) **[6]** Compute the expectation of N.

(b) **[6]** Compute the variance of N.

Exercise 4

Let X and Y be discrete random variables with joint probability mass function given by

$$p_{X,Y}(k,l) = \begin{cases} \frac{1}{2^k n}\binom{k}{l}, & \text{if } k \in \{1,\dots,n\} \text{ and } l \in \{0,\dots,k\} \\ 0 & \text{else,} \end{cases}$$

where $n \in \mathbb{N}$ is fixed.

(a) **[3]** What is the expectation of X?

(b) **[3]** What is the expectation of Y?

(c) **[3]** What is the probability of X being equal to Y?

Exercise 5

A non-negative real-valued random variable X has distribution function

$$F_X(x) = \begin{cases} K \exp\{-x^{-1/8}\}, & \text{for } x > 0, \\ 0 & \text{else,} \end{cases}$$

for some $K > 0$.

(a) **[3]** Find the value of K.

(b) **[4]** What is the probability that $X > (\log 2)^{-8}$?

(c) **[4]** Is X a continuous random variable? In the affirmative case, specify its density function.

Exercise 6

Consider two independent random variables X and Y both distributed as an exponential with parameter 1.

(a) **[9]** Find the joint density function of the random variables U and V given by

$$U = X + Y \quad \text{and} \quad V = \frac{X}{X+Y}.$$

(b) **[4]** Are U and V independent?

(c) **[5]** What is the distribution of V?

Exercise 7

Let $(Z_i)_{i=1}^n$ be a sequence of independent random variables distributed as an exponential with parameter 2. Denote by $S_n = \sum_{i=1}^n Z_i$ the sum of these n i.i.d. random variables.

(a) **[8]** For n large, give an approximate value for the probability that $S_n \geq n/2$.

(b) **[8]** Determine the density function of S_n.

Exercise 8

It is given a Markov chain $(X_n)_{n\geq 0}$ with state space $S = \{-1, 0, +1\}$ and transition matrix $P = (p_{i,j} : i, j \in S)$ such that

$$p_{-1,-1} = p_{-1,0} = p_{+1,-1} = p_{+1,+1} = 1/2, \quad p_{0,-1} = 0 \quad \text{and} \quad p_{0,0} = 0.7.$$

(a) **[2]** Write all the entries of the transition matrix.

(b) **[3]** If $X_0 = -1$, what is the probability that it will still be in state -1 after three steps?

(c) **[4]** Compute the corresponding invariant distribution.

(c) **[5]** If $X_0 = i$, for $i \in S$, is it possible to compute the variance of X_n in the limit $n \to \infty$? In the affirmative case, give its value.