INTRODUCTION TO MEASURE THEORY & INTEGRATION
JULY 12, 2018, 14:00-17:00

- There are 6 questions, worth 100 points.
- You need ≥ 50 points to pass the exam part of the course.
- Clearly state which results you are using without proof.
- Clearly indicate when you switch from Lebesgue integration to Riemann integration or back. Explain very briefly why this is allowed (e.g., function is continuous and bounded, etc).

Question 1 (15 points). Give definition of a σ-algebra, Borel σ-algebra $\mathcal{B}(\mathbb{R})$, and prove that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$, where $\mathcal{C}$ is a collection of all sets of the form $(a, b]$, $a, b \in \mathbb{R}$.

Question 2 (10 points).
(a) Suppose $(\Omega, \mathcal{A})$ is measurable space. Give a definition of a Borel measurable function $f : \Omega \to \mathbb{R}$.
(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a monotonic function. Show that f is Borel measurable.

Question 3 (25 points).
(a) State the Hölder inequality.
(b) Let λ be the Lebesgue measure on $(1, +\infty)$. Show that

$$\int_{(1,+\infty)} \frac{\sqrt[3]{1+x}}{x^2} \lambda(dx) \leq \sqrt[3]{6}$$

(c) Show that the inequality is in fact strict.

Question 4 (10 points). State the Dominated Convergence theorem and prove that

$$\lim_{n\to\infty} \int_{(0,+\infty)} \frac{\sin^n(x)}{x(x+1)} \lambda(dx) = 0.$$

Question 5 (15 points). Let λ be the Lebesgue measure on $[0, 1]$. For $t \geq 0$, let $[t]$ be the integer part of t, i.e., the largest integer m such that $m \leq t$. For each $n \in \mathbb{N}$ let

$$k_n = [\log_2 n], \quad j_n = n - 2^{k_n}, \ A_n = \left[\frac{j_n}{2^{k_n}}, \frac{j_n + 1}{2^{k_n}}\right] \subset [0, 1] \text{ and } \quad f_n(x) = \mathbf{1}_{A_n}(x),$$

Does this sequence converge in measure, almost everywhere, and in $L^1([0, 1], \lambda)$?

Question 6 (25 points). Denote by $\lambda^{(1)}$ the Lebesgue measure on $\mathbb{R}_+ = (0, +\infty)$ and by $\lambda^{(2)}$ the Lebesgue measure on $\mathbb{R}_+^2 = (0, \infty) \times (0, \infty)$.
Suppose $f : \mathbb{R}_+ \to \mathbb{R}$ is a measurable function such that there exists $\alpha \in (0, 1)$ with

$$|f(t)| \leq \frac{t^\alpha}{1+t} \quad \text{for all } t \geq 0.$$

(a) State the Tonelli theorem.
(b) Show that $G(x, t)$ given by

$$G(x, t) = e^{-xt} f(t) \quad (x, t) \in \mathbb{R}_+^2,$$

is integrable, i.e., $G \in L^1(\mathbb{R}_+^2, \lambda^{(2)})$.
(c) State the Differentiation Lemma and prove that if the function $g(t) = t \cdot f(t)$ is Lebesgue integrable on $(0, +\infty)$, then the function $h(x)$, given by

$$h(x) = \int_{(0,+\infty)} e^{-xt} f(t) \ \lambda^{(1)}(dt),$$

is differentiable on $(0, +\infty)$.