INTRODUCTION TO MEASURE THEORY & INTEGRATION
JUNE 19, 2018, 14:00-17:00

- There are 6 questions, worth 100 points.
- You need ≥ 56 points to pass the exam.
- Clearly state which results you are using without proof.
- Clearly indicate when you switch from Lebesgue integration to Riemann integration or back. Explain very briefly why this is allowed (e.g., function is continuous and bounded, etc).

Question 1 (15 points). Let Ω be a set and consider a collection of subsets

$$\mathcal{A} = \{A \subseteq \Omega : \ A \text{ or } A^c \text{ is at most countable}\}.$$

Give definition of a σ-algebra and show that $\mathcal{A}$ is indeed a σ-algebra. For which Ω

$$\text{(i) } \Omega \text{ is finite}, \quad \text{(ii) } \Omega = \mathbb{Q}, \quad \text{(iii) } \Omega = \mathbb{R},$$

does the equality $\mathcal{A} = \mathcal{P}(\Omega)$ hold?

Question 2 (15 points).
(a) Suppose $(\Omega, \mathcal{A})$ is measurable space. Give a definition of a Borel measurable function $f : \Omega \to \mathbb{R}$.
(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Show that f is Borel measurable.

Question 3 (20 points).
(a) State the Hölder inequality.

Let λ be the Lebesgue measure on $\Omega = [0, 1]$. Suppose $\{f_n\}$, $f_n : \Omega \to \mathbb{R}$, is a sequence of Borel measurable functions such that

$$\lim_{n\to\infty} \int_\Omega |f_n|^3 \lambda(dx) = 0.$$

(b) Show that $\lim_{n\to\infty} \int_\Omega |f_n|^p \lambda(dx) = 0$ for all $0 < p < 3$.
(c) Show that

$$\lim_{n\to\infty} \int_\Omega \frac{f_n(x)}{\sqrt{x}} \lambda(dx) = 0.$$

Question 4 (15 points). State the Monotone Convergence theorem and prove that for every $p > 0$ one has

$$\int_{(0,+\infty)} \frac{x}{e^{px}(1 - e^{-x})} \lambda(dx) = \sum_{k=0}^{+\infty} \frac{1}{(k+p)^2}.$$

Question 5 (10 points). Let λ be the Lebesgue measure on $[0, 1]$. Consider the following sequence of functions.

$$f_n(x) = nxe^{-nx^2}, \quad n \geq 1.$$

Does this sequence converge in measure, almost everywhere, and in $L^1([0, 1], \lambda)$.

Question 6 (25 points). Suppose λ is the Lebesgue measure on $\mathbb{R}_+ = [0, \infty)$. Suppose $f, g \in L^1(\mathbb{R}_+, \lambda)$ and f additionally satisfies

$$\int_{\mathbb{R}_+} \frac{|f(t)|}{t} \lambda(dt) < \infty.$$

(a) Show that the function $h(x) = \int_{\mathbb{R}_+} g(xt)f(t)\lambda(dt)$, $x \geq 0$, is integrable, i.e., $h \in L^1(\mathbb{R}_+, \lambda)$.
Hint: Try to prove and use that for any non-negative function k on $\mathbb{R}_+$ and any $c > 0$ one has

$$\int_{\mathbb{R}_+} k(cx)\lambda(dx) = \frac{1}{c} \int_{\mathbb{R}_+} k(x)\lambda(dx).$$

(b) Show that if g is bounded and continuous on $[0, \infty)$, then the function h is bounded and continuous on $[0, \infty)$ as well.