Inleiding kansrekening

Teacher: L. Avena
2nd written examination: Wednesday 4 July 2018, 14:00–17:00.

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation. Formulas alone are not enough. Formulate your answers clearly and carefully.
- The use of textbooks, lecture notes or handwritten notes is not allowed.
- To each question is associated a score written in boldface e.g. in Exercise 1, [3] points are assigned to question (a). Total score: 110. Pass: ≥ 55; no pass: ≤ 54.

Exercise 1 (12 points in total)

A woman possesses 5 coins, 2 of which are double-headed (i.e. head is on both faces), 1 is doubled-tailed, and 2 are normal fair coins. She closes her eyes, picks a coin uniformly at random, and tosses it.

(a) [3] What is the probability that the lower face of the tossed coin is a head?

(b) [4] When she opens her eyes, she sees that the coin is showing a head. What it is the probability that the lower face is a head?

(c) [5] After seeing a head in her first toss, she shuts her eyes and tosses the same coin again. What it is the probability that the lower face is a head?

Exercise 2 (10 points in total)

The life time of an electrical device is an exponential random variable with parameter 1. Consider n independent such devices and let $M_n$ be the first time at which all the devices are not working any longer.

(a) [6] What is the distribution of $M_n$?
(b) [4] Compute the limit, as \( n \) goes to infinity, of \( P(M_n > \log n) \).

**Exercise 3** (10 points in total)

One hundred cars are participating to a challenging race competition where each car has chance 1/2 to reach the end of the race. For each car reaching the end of the race, the official photographer of the competition will take a number of pictures distributed as a Poisson random variable with parameter 2.

(a) [10] What is the expectation of the total number of pictures at the end of the race?

**Exercise 4** (15 points in total)

10 people arrive (independently of each other) at the post office at a time uniformly distributed between 8:01 and 9:00 o’clock. Let \( X \) be the number of people arriving in the first 15 minutes and \( Y \) be the number of people arriving in the last 15 minutes.

(a) [6] Compute expectation and variance of \( X \)

(b) [9] Compute covariance and correlation of \( X \) and \( Y \).

**Exercise 5** (11 points in total)

Consider two real valued continuous random variables with joint density function

\[
f_{X,Y}(x, y) = \begin{cases} 
K(x + 2y), & \text{for } (x, y) \in [0, 1] \times [0, 1], \\
0, & \text{else.}
\end{cases}
\]

for some \( K \in \mathbb{R} \).

(a) [3] Find the value of \( K \).

(b) [4] What is the probability that \( X > 1/2 \)?

(c) [4] Are \( X \) and \( Y \) independent?

**Exercise 6** (10 points in total)

Let \( Q \) be a point uniformly distributed in a square with sides of length 1. Denote by \( D \) the (euclidian) distance of the point \( Q \) from the center of the square.
Compute the expectation of $D^2$.

(Hint: center the square at the origin of the x-y axis, and express $D^2$ in terms of the coordinates $X$ and $Y$ of $Q = (X, Y)$)

**Exercise 7** (24 points in total)

A die is thrown $n$ (independent) times. For $i \leq n$, define the random variables

$$Y_i = \begin{cases} 
0, & \text{if } X_i \text{ is odd}, \\
\frac{X_i}{2}, & \text{if } X_i \text{ is even}, 
\end{cases}$$

where $X_i$ denotes the outcome of the die at the $i$-th thrown. Consider the sum $S_n = \sum_{i=1}^{n} Y_i$

(a) [8] Compute expectation and variance of $S_n$.

(b) [8] What can you say about the limit, as $n$ goes to infinity, of $P\left(\frac{S_n-n}{n} > 0.1\right)$?

(c) [8] What can you say about the limit, as $n$ goes to infinity, of $P\left(\frac{S_n-n}{\sqrt{n}} > 0.1\right)$?

**Exercise 8** (18 points in total)

For $c \in \mathbb{R}$, consider the 3 by 3 matrix:

$$P = \begin{bmatrix} 
0 & 1 & 0 \\
c & 0 & 1/2 \\
1 & 0 & 0 
\end{bmatrix}$$

(a) [2] For which values of $c$ is $P$ a transition matrix?

(b) [5] When $P$ is a transition matrix, what can be said about the period of the three corresponding states?

(c) [5] When $P$ is a transition matrix, let $X = (X(n))_{n \geq 0}$ be the associated Markov chain and call $L$ the state corresponding to the 3rd row of $P$. Compute $P(X(3k) = L | X(0) = L)$ for arbitrary $k \in \mathbb{N}$.

(d) [6] On the long run of the chain in point (c), is there a state that has the tendency to be visited less than the others regardless of the starting point?