# Inleiding kansrekening

*Teacher:* L. Avena
*2nd written examination:* Wednesday 4 July 2018, 14:00–17:00.

- Write your name and student identification number **on each piece of paper** you hand in.
- All answers **must come with a full explanation**. Formulas alone are not enough. Formulate your answers clearly and carefully.
- The use of textbooks, lecture notes or handwritten notes **is not allowed**.
- To each question is associated a score written in boldface e.g. in **Exercise 1**, [**3**] points are assigned to question (a). *Total score:* 110. Pass: $\geq 55$; no pass: $\leq 54$.

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**Exercise 1** (12 points in total)

A woman possesses 5 coins, 2 of which are double-headed (i.e. head is on both faces), 1 is doubled-tailed, and 2 are normal fair coins. She closes her eyes, picks a coin uniformly at random, and tosses it.

(a) [**3**] What is the probability that the lower face of the tossed coin is a head?

(b) [**4**] When she opens her eyes, she sees that the coin is showing a head. What it is the probability that the lower face is a head?

(c) [**5**] After seeing a head in her first toss, she shuts her eyes and tosses the same coin again. What it is the probability that the lower face is a head?

**Exercise 2** (10 points in total)

The life time of an electrical device is an exponential random variable with parameter 1. Consider $n$ independent such devices and let $M_n$ be the first time at which all the devices are not working any longer.

(a) [**6**] What is the distribution of $M_n$?

(b) **[4]** Compute the limit, as $n$ goes to infinity, of $P(M_n > \log n)$ .

**Exercise 3** (10 points in total)

One hundred cars are participating to a challenging race competition where each car has chance $1/2$ to reach the end of the race. For each car reaching the end of the race, the official photographer of the competition will take a number of pictures distributed as a Poisson random variable with parameter 2.

(a) **[10]** What is the expectation of the total number of pictures at the end of the race?

**Exercise 4** (15 points in total)

10 people arrive (independently of each other) at the post office at a time uniformly distributed between 8:01 and 9:00 o'clock. Let $X$ be the number of people arriving in the first 15 minutes and $Y$ be the number of people arriving in the last 15 minutes.

(a) **[6]** Compute expectation and variance of $X$

(b) **[9]** Compute covariance and correlation of $X$ and $Y$.

**Exercise 5** (11 points in total)

Consider two real valued continuos random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} K(x+2y), & \text{for } (x,y) \in [0,1] \times [0,1], \\ 0 & \text{else.} \end{cases}$$

for some $K \in \mathbb{R}$.

(a) **[3]** Find the value of $K$.

(b) **[4]** What is the probability that $X > 1/2$?

(c) **[4]** Are $X$ and $Y$ independent?

**Exercise 6** (10 points in total)

Let $Q$ be a point uniformly distributed in a square with sides of length 1. Denote by $D$ the (euclidian) distance of the point $Q$ from the center of the square.

(a) **[10]** Compute the expectation of $D^2$.
*(Hint: center the square at the origin of the $x$-$y$ axis, and express $D^2$ in terms of the coordinates $X$ and $Y$ of $Q = (X, Y)$)*

**Exercise 7** (24 points in total)

A die is thrown $n$ (independent) times. For $i \leq n$, define the random variables

$$Y_i = \begin{cases} 0, & \text{if } X_i \text{ is odd,} \\ \frac{X_i}{2} & \text{if } X_i \text{ is even,} \end{cases}$$

where $X_i$ denotes the outcome of the die at the $i$-th thrown. Consider the sum $S_n = \sum_{i=1}^{n} Y_i$

(a) **[8]** Compute expectation and variance of $S_n$.

(b) **[8]** What can you say about the limit, as $n$ goes to infinity, of $P(\frac{S_n - n}{n} > 0.1)$?

(c) **[8]** What can you say about the limit, as $n$ goes to infinity, of $P(\frac{S_n - n}{\sqrt{n}} > 0.1)$?

**Exercise 8** (18 points in total)

For $c \in \mathbb{R}$, consider the 3 by 3 matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ c & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) **[2]** For which values of $c$ is $P$ a transition matrix?

(b) **[5]** When $P$ is a transition matrix, what can be said about the period of the three corresponding states?

(c) **[5]** When $P$ is a transition matrix, let $X = (X(n))_{n \geq 0}$ be the associated Markov chain and call $L$ the state corresponding to the 3rd row of $P$. Compute $P(X(3k) = L | X(0) = L)$ for arbitrary $k \in \mathbb{N}$.

(d) **[6]** On the long run of the chain in point (c), is there a state that has the tendency to be visited less than the others regardless of the starting point ?