Resit examination: Curves over Finite Fields
Monday 15th April 2019

This is an open book exam. Justify your answers and explain your working. You may use the results of earlier parts of questions in later parts, even if you have not answered them. Electronic devices including calculators and phones are **not** permitted.

1. Let $C \subset \mathbb{A}^2(\mathbb{C})$ be the smooth affine curve given by the equation

$$x^4 + y^4 = 1.$$

For $r = 0, 1, 2, 3$, let P_r be the point $(i^r, 0) \in C$.

(a) Show that y is a local parameter at each of P_0, P_1, P_2, P_3.

(b) Let ω be the differential dx on C. Show

$$\mathrm{div}(\omega) = 3(P_0 + P_1 + P_2 + P_3).$$

(c) Let $\bar{C}$ be the projective closure of C. Show that $\bar{C}$ has precisely four points Q_0, Q_1, Q_2, Q_3 at infinity.

(d) Show that $\mathrm{ord}_{Q_r}(\omega) = -2$ for $r = 0, 1, 2, 3$, and hence compute the genus of C.

2. Let $C \subset \mathbb{A}^2(\mathbb{F}_5)$ be the affine curve defined by the equation

$$y^3 = x^4 + 1.$$

(a) Show that the projective closure $\bar{C}$ of C is smooth and has a single point at infinity.

(b) Explain briefly why, for a finite field $\mathbb{F}_q$ with $q \equiv 2 \pmod 3$, the map $x \mapsto x^3$ is a bijection on $\mathbb{F}_q$.

(c) Deduce that $\#\bar{C}(\mathbb{F}_5) = 6$ and $\#\bar{C}(\mathbb{F}_{5^3}) = 126$.

(d) Given that $\bar{C}$ has genus 3 and that $\#\bar{C}(\mathbb{F}_{5^2}) = 20$, compute the zeta function of $\bar{C}$ over $\mathbb{F}_5$.

The exam continues on the back of this sheet.

3. Consider the following list of rational functions.

The list contains the zeta functions of each of the following curves over the field $\mathbb{F}_7$. For each curve, identify its zeta function. Justify your answers.

$$Z_1 = \frac{1+2T+7T^2}{(1-T)(1-7T)} \qquad Z_4 = \frac{1+2T+14T^3+49T^4}{(1-T)(1-7T)}$$

$$Z_2 = \frac{1}{1-7T} \qquad Z_5 = \frac{1-9T-63T^3+49T^4}{(1-T)(1-7T)}$$

$$Z_3 = \frac{1-3T+7T^2}{(1-T)(1-7T)} \qquad Z_6 = \frac{1-T}{1-7T}$$

(a) A smooth projective curve C_1 of genus 1 satisfying $\#C(\mathbb{F}_7) = 5$;

(b) a smooth projective curve C_2 of genus 1 satisfying $\#C(\mathbb{F}_7) = 10$;

(c) a smooth projective curve C_3 of genus 2;

(d) the smooth affine curve defined by $x^2 - y^2 = 1$.