EXAM Introduction to Dynamical Systems

Wednesday January 23, 2019, 10:00 - 13:00 h.

Write your name and student number on each piece of paper you hand in.

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1. Consider, for $\varepsilon \in \mathbb{R}$ with $|\varepsilon|$ sufficiently small and $f : \mathbb{R}^2 \to \mathbb{R}$ sufficiently smooth with $f(0,0) = 0$, the planar system,

$$\ddot{x} + \varepsilon f(x, \dot{x}) + x - x^3 = 0 \;\text{ or, equivalently, }\; \left\{ \begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & -x + x^3 - \varepsilon f(x,y) \end{array} \right. \tag{1}$$

Note that system (1) has 3 critical points that do not depend on ε, $(-1,0)$, $(0,0)$ and $(1,0)$.

Part **1A**: $\varepsilon = 0$.

1A(i). Determine the integral $H(x,y)$ of the integrable system (1) (with $\varepsilon = 0$).

1A(ii). Let $(x_{\rm p}(t), y_{\rm p}(t))$ be the solution of (1) with $(x_{\rm p}(0), y_{\rm p}(0)) = (\frac{1}{2}, 0)$: determine $H_p = H(x_{\rm p}, y_{\rm p})$ and show that $(x_{\rm p}(t), y_{\rm p}(t))$ is periodic.

1A(iii). Give a sketch of the phase portrait of (1) (with $\varepsilon = 0$); indicate explicitly the orbit $(x_{\rm p}(t), y_{\rm p}(t))$ in this sketch.

Part **1B**: $\varepsilon > 0$, sufficiently small, $f(x,y) = y$.

1B(i). Show that $\dot{H}(x,y) \leq 0$ (with $H(x,y)$ as found in **1A**(i)).

1B(ii). Now consider the solution $(x_{\rm fric}(t), y_{\rm fric}(t))$ of (1) with (again) $(x_{\rm fric}(0), y_{\rm fric}(0)) = (\frac{1}{2}, 0)$: show that $\lim_{t\to\infty}(x_{\rm fric}(t), y_{\rm fric}(t)) = (0,0)$; give a sketch.

1B(iii). Give a sketch of the basin of attraction $W^s((0,0))$ of the (critical) point $(0,0)$ in which you clearly indicate its boundary $\partial W^s((0,0))$.

1B(iv). List all possible omega-limit sets $\omega(x_0, y_0)$ associated to (1) – where (x_0, y_0) is the initial condition of the solution $(x_{\rm fric}(t), y_{\rm fric}(t))$ of (1) (with $\varepsilon > 0$ and $f(x,y) = y$).

Part **1C**: $f(x,y) = y\,(H(x,y) - H_p)$, with $H(x,y), H_p$ as in **1A**(i),(ii), $|\varepsilon|$ sufficiently small.

1C(i). Show that $(x_{\rm p}(t), y_{\rm p}(t))$ as defined in **1A**(ii) (also) is a periodic solution of (1) if $\varepsilon \neq 0$.

1C(ii). Set $(x,y) = (x_p(t) + \xi, y_p(t) + \eta)$ and determine the linearized 2-dimensional (time-periodic) system for (ξ, η).

1C(iii). Let λ_1 and λ_2 be the Floquet exponents associated to the linear system determined in **1C**(ii): note that $\lambda_1 = 0$ and derive an analytical expression for λ_2. Discuss the stability of $(x_{\rm p}(t), y_{\rm p}(t))$ as function of ε.

See Reverse Side

2. Consider for $g, h : \mathbb{R}^2 \to \mathbb{R}$ sufficiently smooth the 2-dimensional system,

$$\left\{ \begin{array}{rcl} \dot{x} & = & -x \\ \dot{y} & = & g(x)y + h(x)y^3 \end{array} \right. \tag{2}$$

Note that $(0,0)$ is a critical point of system (2) for all (smooth) $f(x)$ and $g(x)$ (and that it is not necessarily the only critical point of (2)).

Part **2A**: the stability of $(0,0)$.

2A(i). Show that $(0,0)$ is stable if $g(0) < 0$ and unstable if $g(0) > 0$.

2A(ii). Let $g(0) = 0$: show that the center manifold $W^c((0,0))$ of $(0,0)$ is *exactly* given by the y-axis, i.e. that $W^c((0,0)) = \{(x,y) \in \mathbb{R}^2 : x = 0\}$.

2A(iii). Let $g(0) = 0$ and assume $h(0) \neq 0$: what can you say about the (asymptotic) stability of $(0,0)$?

2A(iv). Let $g(0) = h(0) = 0$: what can you say about the (asymptotic) stability of $(0,0)$? Give a sketch of the flow generated by (2), i.e. its (local) phase portrait, in a neighborhood of $(0,0)$.

Part **2B**: $g(x) \equiv 0$ – explicit solutions and finite-time blow-up.

2B(i). Let $(x(t;(x_0,y_0)), y(t;(x_0,y_0)))$ be the solution of (2) with initial condition (x_0,y_0): determine an explicit expression for $(x(t;(x_0,y_0)), y(t;(x_0,y_0)))$ and show that it is given by,

$$x(t;(x_0,y_0)) = x_0 e^{-t}, \quad y(t;(x_0,y_0)) = \frac{y_0}{\sqrt{1 - 2y_0^2 \int_0^t h(x_0 e^{-s})\, ds}}$$

(recall that $g(x) \equiv 0$).

2B(ii). Assume that $h(0) > 0$: show that $(x(t;(x_0,y_0)), y(t;(x_0,y_0)))$, or more specifically its y-component $y(t;(x_0,y_0))$, necessarily blows up in finite time.

2B(iii). Assume that $h(0) < 0$: show that $\lim_{t\to\infty}(x(t;(x_0,y_0)), y(t;(x_0,y_0))) = (0,0)$ for $|x_0|$ sufficiently small.

2B(iv). Let $h(0) = 0$ and define,

$$\mathcal{H}_\infty(x_0) = \int_0^\infty h(x_0 e^{-s})\, ds.$$

Note that solutions $(x(t;(x_0,y_0)), y(t;(x_0,y_0)))$ must again blow up in finite time if $1 - 2y_0^2 \mathcal{H}_\infty(x_0) < 0$ (by an argument similar to that of **2B**(ii)). Can solutions blow up in finite time if $1 - 2y_0^2 \mathcal{H}_\infty(x_0) > 0$?

2B(v). Keep $h(0) = 0$: show that solutions $(x(t;(x_0,y_0)), y(t;(x_0,y_0)))$ exist for all $t \geq 0$ if x_0 is sufficiently small.

2B(vi). Continue with $h(0) = 0$ and assume that $\mathcal{H}_\infty(x_0) > 0$: determine the limit behavior $\lim_{t\to\infty}(x(t;(x_0,y_0)), y(t;(x_0,y_0)))$ as function of (x_0,y_0). Give a sketch of the phase portrait.