Linear Analysis

24 January 2019 14:00-17:00hr

- If you are not a native English speaker and you are in doubt about the meaning of the questions, please ask the invigilator.
- You can answer the questions in English or in Dutch. If your choice is Dutch, please feel free to use English terminology when convenient.
- You can use the results of the earlier parts of a question, even if you have not solved these parts.
- The point distribution is preliminary and may be subject to change.
- This exam has four questions on two pages.

---

1. Let X and Y be normed spaces, and let $B(X, Y)$ denote the normed space of all bounded linear operators from X into Y. Suppose that f is a continuous linear functional on X and that y is an element of Y, and define $T_{f,y} : X \to Y$ by setting

$$T_{f,y}(x) := f(x)y$$

for $x \in X$. Then $T_{f,y}$ is obviously linear—this need not be shown.

3 pt. (a) Show that T is a *bounded* linear map.

3 pt. (b) Show that $\|T_{f,y}\| = \|f\|\|y\|$.

3 pt. (c) Suppose that $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. Show that $\{T_{f,y_n}\}_{n=1}^{\infty}$ is a Cauchy sequence in $B(X, Y)$.

3 pt. (d) Suppose that $X \neq \{0\}$. Then there exist an element x of X and a continuous linear functional f on X that such $f(x) = 1$. Why?

4 pt. (e) Suppose that $X \neq \{0\}$ and that $B(X, Y)$ is a Banach space. Show that Y is a Banach space.

8 pt. 2. Let M_1 and M_2 be non-empty compact metric spaces. For $i = 1, 2$, let $C(M_i, \mathbb{R})$ be the Banach space of all continuous real-valued functions on M_i, supplied with the usual norm $\|\cdot\|_\infty$ that is defined by

$$\|f\|_\infty := \max\{\, |f(x)| : x \in M_i \,\}$$

for $f \in C(M_i, \mathbb{R})$.
Suppose that $T : C(M_1, \mathbb{R}) \to C(M_2, \mathbb{R})$ is a linear map with the following property: if $\{f_n\}_{n=1}^{\infty}$ is a sequence in $C(M_1, \mathbb{R})$ such that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in M_1$, then $\lim_{n\to\infty} Tf_n(x) = 0$ for all $x \in M_2$. Show that T is continuous.

Exam continues overleaf

3. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be elements of the *complex* Hilbert space ℓ^2.

4 pt. (a) Show that the sequence

$$(x_1 y_1, x_1 y_1 + x_2 y_2, x_1 y_1 + x_2 y_2 + x_3 y_3, \ldots)$$

is an element of ℓ^∞.

4 pt. (b) Show that the sequence

$$\left(\frac{x_1 y_1}{1}, \frac{x_1 y_1 + x_2 y_2}{2}, \frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{3}, \ldots \right)$$

is an element of ℓ^2.

Fix $x = (x_1, x_2, \ldots) \in \ell^2$, and define $T_x : \ell^2 \to \ell^2$ by setting

$$T_x(y) := \left(\frac{x_1 y_1}{1}, \frac{x_1 y_1 + x_2 y_2}{2}, \frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{3}, \ldots \right)$$

for $y \in \ell^2$. Then T_x is obviously linear—this need not be shown.

3 pt. (c) Show that T_x is a *bounded* linear map.

4 pt. (d) For $n = 1, 2, \ldots$, let e_n denote the n^{th} standard basis vector of ℓ^2, with 1 in the n^{th} spot and zeroes elsewhere. Compute $T_x^* e_n$, where T_x^* is the Hilbert space adjoint of T_x.

4. Let c_0 be the space of all real sequences converging to zero, supplied with usual norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty := \max_{n \geq 1} |x_n|$$

for $x = (x_1, x_2, \cdots) \in c_0$. Then c_0 is a Banach space. As shown in the book and during the lectures, however, it is not reflexive. This question will provide an alternative proof of the latter fact.

Define $f : c_0 \to \mathbb{R}$ by setting

$$f(x) := \sum_{n=1}^{\infty} \frac{x_n}{n!}$$

for $x = (x_1, x_2, \cdots) \in c_0$. It is clear that f is a linear functional—this need not be shown.

3 pt. (a) Show that f is a *bounded* linear functional.

3 pt. (b) Show that $\|f\| = \mathrm{e} - 1 \left(= \sum_{n=1}^{\infty} \frac{1}{n!}\right)$.

3 pt. (c) Show that $|f(x)| < \mathrm{e} - 1$ for all $x \in c_0$ such that $\|x\| = 1$.

It is then clear from combining the parts (b), (c), and the following part (d) that c_0 is not reflexive.

5 pt. (d) Let X be a non-zero reflexive space with dual space X'. Suppose that $f \in X'$. Show that there exists an $x \in X$ with $\|x\| = 1$ such that $\|f\| = f(x)$.

Preliminary point distribution

Question:	1	2	3	4	Total
Points:	16 (3+3+3+3+4)	8 (8)	15 (4+4+3+4)	14 (3+3+3+5)	53

Grade := (total number of points)/5.3