Examination for the course on

Random Walks

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Wednesday, January 9, 2019, 14:00–17:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or lecture notes is not allowed.
- There are 8 problems. The total number of points is 100 (per question indicated in boldface). A score of ≥ 50 points is necessary to pass the exam part. The final grade is the weighted average (75%) exam and (25%) homework grades

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(1) Consider a simple random walk $(S_n)_{n=0,\dots,N}$ on $\mathbb{Z}$.

- (a) **[5]** Give definition of a stopping time.
- (b) **[5]** Suppose T is a stopping time, prove that $\{T \geq k\}^c \in A_{k-1}$ for all $k \geq 1$, where $\{A_k\}$ is the filtration of observable events.

(2) **[10]** Denote by $\{S_n\}$ and $\{\tilde{S}_n\}$ are two independent random walks on $\mathbb{Z}$. Show that

$$S_n^{(2)} = \left(\frac{S_n + \tilde{S}_n}{2}, \frac{S_n - \tilde{S}_n}{2} \right)$$

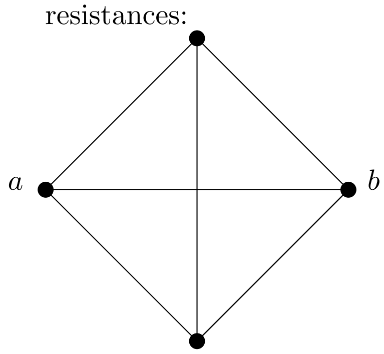
is a simple random walk on the lattice $\mathbb{Z}^2$.

(3) Consider a simple random walk $(S_n)_{n=0}^{\infty}$ on $\mathbb{Z}^d$.

(a) **[5]** Define a notion of recurrence of a random walk on the lattice $\mathbb{Z}^d$. and formulate a criterion for recurrence in terms of the random walk Green function $G_d(x; z)$.

(b) **[10]** Prove that $G_d(x; 1)/G_d(0; 1) > 0$ for all $x \in \mathbb{Z}^d$, and formulate a criterion for recurrence of random walk in terms $G_d(x, 1)$.

(4) **[5]** Compute the effective resistance between a and b of the following network of unit resistances:



[5] What will happen with the the effective resistance between a and b, if the direct link between a and b is removed? Formulate the general principle.

(5) Suppose A is a finite subset of $\mathbb{Z}^2$. Denote by ∂A the external boundary of A:

$$\partial A = \{y \in \mathbb{Z}^2 \setminus A : \quad \exists x \in A : ||x - y|| = 1\}.$$

Denote by $\{S_n^x\}$ the simple random walk, which starts at $x \in A \cup \partial A$: $S_0^x = x$. Let

$$\tau_x = \inf\{j \geq 0 : S_j^x \in \partial A\}.$$

(a) **[5]** Show that $\tau_x = 0$ if $x \in \partial A$, and positive otherwise.

Suppose g is a real-valued function defined on the boundary ∂A; i.e., $g : \partial A \to \mathbb{R}$.

(b) **[15]** Prove that

$$f(x) = \mathbb{E} g(S_{\tau_x}^x)$$

is the unique harmonic function on A, such that

$$f(x) = g(x) \quad \forall x \in \partial A.$$

(6) Consider a one-dimensional simple random walk $\{S_n\}$, $S_0 = 0$.
(a) **[5]** Prove that $\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(\sigma_{-1} > 2n - 1) = \mathbb{P}(\sigma_1 > 2n - 1)$.

(b) **[5]** Using the fact that $\mathbb{P}(\sigma_a \leq n) = \mathbb{P}(S_n \notin [-a, a - 1])$ for all $a, n \geq 1$, deduce from the previous statement that

$$\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(S_{2n} = 0).$$

(7) Brownian motion.

(a) **[10]** Show that, for every point $x \in \mathbb{R}$, there exists a two-sided Brownian motion starting x, i.e., $\{W(t) : t \in \mathbb{R}\}$ with $W(0) = x$, which has continuous paths, independent increments and the property that, for all $t \in \mathbb{R}$ and $h > 0$, the increments $W(t + h) - W(t)$ are normally distributed with expectation zero and variance h.

(b) **[5]** Let $(W(t))_{t \geq 0}$ and $(\widetilde{W}(t))_{t \geq 0}$ be independent standard Brownian motions on $\mathbb{R}$. Is the process

$$\widehat{W}_t = \frac{W_t - \widetilde{W}_t}{\sqrt{2}},$$

again a standard Brownian motion?

(8) Suppose that the current price of a stock is $S_0 = 100$ euro, and that at the end of a single period of time its price is either $S_1 = 85$ euro or $S_1 = 135$ euro. A trading company X offers a new financial instrument (derivative) called 'double your profit': if the price goes up after one period, the company will pay you $2 \times 35 = 70$ euros, and if the price goes down, you have to pay $2 \times 15 = 30$ euros.

(a) **[5]** Compute the arbitrage-free price of this derivative with the help of the Binomial Asset Pricing Model. The interest rate is 10%

(b) **[5]** Suppose the company X is prepared to sell to you one such derivative for 1 euro less than the the arbitrage-free price you have just determined. What is your course of action and eventual profit?