Exam introduction to differentiable manifolds 1, 1-15-2019
Always motivate your answers and state the theorems/results you are using.

**Question 1**
For fixed real number $q > 0$ consider the following system of equations:

\[
q^x + q^y + q^z = 2q + q^q \\
-x^q + y^q + z^q = q^q
\]

a. Prove the following statement about the set $S$ of solutions to the above equations: When $q \neq 1$, there is an open neighborhood $U \subset \mathbb{R}^3$ of the point $(1, 1, q) \in S$ such that $U \cap S$ is $C^1$ diffeomorphic to an open interval of $\mathbb{R}$.

b. Formulate and prove a similar statement in the case $q = 1$.

**Question 2**

a. Explain how the formula $ydx$ can be interpreted as a $C^1$, 1-covector field $\omega$ on $\mathbb{R}^2$.

b. Express $d\omega$ as a wedge product of two 1-covector fields.

c. Suppose $\gamma : [0, 1] \to \mathbb{R}^2$ is the 1-cube defined by $\gamma(t) = (t, -(t - 1)t)$. Calculate the integral $\int_\gamma \omega$ directly from the definition.

d. Calculate the integral $\int_\gamma \omega$ using Stokes theorem for 2-chains.

**Question 3**

a. Consider a metric $g$ on open set $P \subset \mathbb{R}^n$ and an isometry $\phi : P \to P$. If $\gamma$ is a differentiable curve of minimal length between points $p, q \in P$, show that $\phi \circ \gamma$ is also a differentiable curve of minimal length between $\phi(p), \phi(q)$.

b. Find an element of $Z \in \Lambda^2(\mathbb{R}^4)$ such that $Z \wedge Z \neq 0$ and $\star Z = Z$ with respect to the standard orientation and Euclidean metric.

d. Can you find a function $f : M \to TM$ and find an $\omega$ as in part c. so that $f^* \omega$ is not everywhere zero?

**Question 4**
Consider the manifold $M$ with atlas defined by charts $M^1 = (0, 1)$ and $M^2 = (1, 2)$ and $M^1_2 = (0, 1) - \{\frac{1}{2}\}$ and $M^2_1 = (1, 2) - \{\frac{3}{2}\}$ and transition map

\[
\tau_2^1 : M^1_2 \to M^2_1 \text{ given by } \tau_2^1(t) = \begin{cases} 
  t + \frac{3}{2} & \text{if } t < \frac{1}{2}, \\
  t + \frac{1}{2} & \text{if } t > \frac{1}{2}.
\end{cases}
\]

a. What is the dimension of $M^2$ and is $M$ a $C^2$ manifold?

b. Give an atlas for the tangent bundle $TM$ of $M$.

c. Write down an explicit example of a $C^1$ differentiable 2-covector field $\omega$ on $TM$ that is not everywhere zero.

d. Can you find a function $f : M \to TM$ and find an $\omega$ as in part c. so that $f^* \omega$ is not everywhere zero?