Exam introduction to differentiable manifolds 1, 1-15-2019

Always motivate your answers and state the theorems/results you are using.

Question 1

For fixed real number $q > 0$ consider the following system of equations:

$$q^x + q^y + q^z = 2q + q^q$$
$$-x^q + y^q + z^q = q^q$$

a. Prove the following statement about the set S of solutions to the above equations: When $q \neq 1$, there is an open neighborhood $U \subset \mathbb{R}^3$ of the point $(1, 1, q) \in S$ such that $U \cap S$ is C^1 diffeomorphic to an open interval of $\mathbb{R}$.

b. Formulate and prove a similar statement in the case $q = 1$.

Question 2

a. Explain how the formula ydx can be interpreted as a C^1, 1-covector field ω on $\mathbb{R}^2$.

b. Express $d\omega$ as a wedge product of two 1-covector fields.

c. Suppose $\gamma : [0, 1] \to \mathbb{R}^2$ is the 1-cube defined by $\gamma(t) = (t, -(t-1)t)$. Calculate the integral $\int_\gamma \omega$ directly from the definition.

d. Calculate the integral $\int_\gamma \omega$ using Stokes theorem for 2-chains.

Question 3

a. Consider a metric g on open set $P \subset \mathbb{R}^n$ and an isometry $\phi : P \to P$. If γ is a differentiable curve of minimal length between points $p, q \in P$, show that $\phi \circ \gamma$ is also a differentiable curve of minimal length between $\phi(p), \phi(q)$.

b. Find an element of $Z \in \Lambda^2(\mathbb{R}^4)$ such that $Z \wedge Z \neq 0$ and $\star Z = Z$ with respect to the standard orientation and Euclidean metric.

Question 4

Consider the manifold M with atlas defined by charts $M^1 = (0, 1)$ and $M^2 = (1, 2)$ and $M^1_2 = (0, 1) - \{\frac{1}{2}\}$ and $M^2_1 = (1, 2) - \{\frac{3}{2}\}$ and transition map

$\tau^1_2 : M^1_2 \to M^2_1$ given by $\tau^1_2(t) = \begin{cases} t + \frac{3}{2} \text{ if } t < \frac{1}{2} \\ t + \frac{1}{2} \text{ if } t > \frac{1}{2} \end{cases}$.

a. What is the dimension of M? and is M a C^2 manifold?

b. Give an atlas for the tangent bundle TM of M.

c. Write down an explicit example of a C^1 differentiable 2-covector field ω on TM that is not everywhere zero.

d. Can you find a function $f : M \to TM$ and find an ω as in part c. so that $f^*\omega$ is not everywhere zero?