Name: ______________________________

Final Exam for Differentiable Manifolds 1

Jan 25, 2023

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

- You have three hours for this exam.
- Please show **ALL** your work on your solutions. Partial credit will be awarded where appropriate.
- You may quote without proof the theorems/propositions/lemmas given during lecture as long as you state each result clearly. Points may be deducted for incorrect or missing hypotheses.
- **NO** calculators, or computers are permitted.
- However, you can use any written material: books and notes.
- **NO** form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

This exam will be solely undertaken by myself, without any assistance from others and without use of sources other than those explicitly allowed by the lecturer. Moreover, I have read and signed the declaration of integrity.

Name (printed): ______________________________

Signature: ______________________________

Date: ______________________________

1. Prove or find a counter-example for the following statement:

For any positive integer n and for any pair of finite dimensional vector spaces V and W, there exists an isomorphism of vector spaces

$$\wedge^n(V \oplus W) \to \wedge^n V \oplus \wedge^n W.$$

Sketch of a solution: The statement is in general false. Compare the dimensions of the vector spaces of both sides.

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2. We say that two Frenet curves $\alpha : (-\infty, \infty) \to \mathbb{R}^2$ and $\beta : (-\infty, \infty) \to \mathbb{R}^2$ parametrized by arc lengths are perpendicular if the first Frenet vectors $e_1^\alpha(s)$ and $e_1^\beta(s)$ are perpendicular for every $s \in (-\infty, \infty)$. Show that there exists a rotation A by 90° (either clock-wise or counter clock-wise) and a translation v bringing the traces of the curves together, i.e

$$A\alpha(s) + v = \beta(s)$$

for all $s \in (-\infty, \infty)$.

Sketch of a solution: First, the 90° rotation is consistent, either it is always clock-wise or always counter clock-wise. Indeed, the ratio between the rotated $e_1^\alpha(s)$ and $e_1^\beta(s)$ is ± 1, but it needs to be continuous, so it is either always 1 or -1.

Second, let's rotate α with the correct rotation. Call this curve γ. Then, $e_1^\gamma(s)$ and $e_1^\beta(s)$ coincide for all s. Now, use the Fundamental theorem of the local theory of curves.

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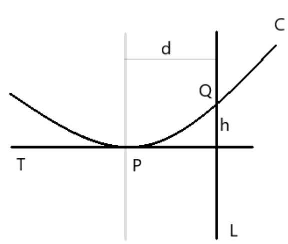
3. Let $\gamma : (-\infty, \infty) \to \mathbb{R}^2$ be a Frenet curve in the plane with trace C. Let T be the tangent line at the point $P = \gamma(0) \in C$. Draw a line L parallel to the normal vector at P of distance d from P (see the figure below). Let Q be the intersection point of C and L "closest" to P, i.e $Q \in C \cap L$, so that $Q = \gamma(s)$ for the smallest $|s|$ possible. Let h be the distance from T to the point Q. Prove that

$$|\kappa(P)| = \lim_{d \to 0} \frac{2h}{d^2}$$

where $\kappa(P)$ is the curvature of C at P.



Sketch of a solution: Consider the Taylor expansion of γ at 0:

$$\gamma(s) = \gamma(0) + \gamma'(0)s + \frac{1}{2}\gamma''(0)s^2 + ...$$

Recall that $\gamma'(0) = (1, 0)$ and $\gamma''(0) = \kappa(P)(0, 1)$, so at the point $s = (d, h)$, we have

$$(d, h) = (s, 0) + \frac{1}{2}\kappa(P)(0, s^2) + ...$$

implying the statement.

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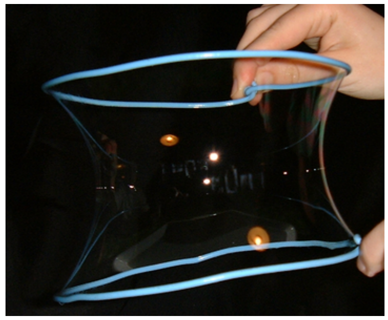
4. Consider the catenoid defined as the image of

$$x(u, v) = (\cos u \cdot \cosh v, \sin u \cdot \cosh v, v)$$

where $u \in [0, 2\pi]$ and $v \in \mathbb{R}$.

(See picture, stretching soap between two parallel circles generates a catenoid)

Show that the catenoid is a regular surface. (Here $\cosh v = \frac{e^v + e^{-v}}{2}$)

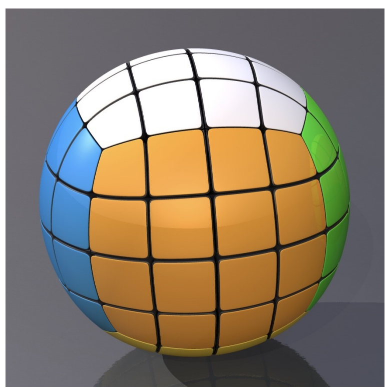


Sketch of a solution: The problem has two parts. First, show that $x_u \times x_v$ is never the 0 vector. Second, show that the catenoid can be covered by two open regular surfaces, for instance a) when $0 < u < 2\pi$, b) when $-\pi < u < \pi$.

5. Show that the catenoid is a minimal surface meaning that its mean curvature is constant 0.
Hint: You can use the following rules: $\frac{d}{dv}\cosh v = \sinh v$, $\frac{d}{dv}\sinh v = \cosh v$ and $\cosh^2 v - \sinh^2 v = 1$.

Sketch of a solution: Compute E, F, G and e, f, g.

6. A square on a sphere is a collection of 4 vertices and 4 arcs connecting consecutive points so that the arcs are parts of great circles, and the arcs do not intersect, and moreover, all angles and all sides are the same. A cube on the sphere is a covering of the sphere with squares so that at every vertex three squares meet, and squares meet exactly at arcs connecting consecutive vertices (see picture). Prove that every cube on the sphere has the same number of faces, and compute this number.



Sketch of a solution: Each angle of a square is $120°$. Therefore, if one draws a diagonal (using a great arc) of a square, it cuts the square into two triangles of angles $120°$, $60°$ and $60°$. From this, using Gauss-Bonnet, you can compute the area of one triangle, and thus, the number of triangles.

7. Consider the set $S = \{(x, y, u, v) : x^4 + y^4 + u^4 + v^4 = 1\} \subset \mathbb{R}^4$.

- Show that this is a three-dimensional topological manifold with the subspace topoogy.
- Find an explicit atlas so that the transition functions are C^1 providing S with a C^1-differentiable manifold structure.

Sketch of a solution: For the first part, we see that $S = f^{-1}(0)$ where $f : \mathbb{R}^4 \to \mathbb{R}$ is the function $(x, y, u, v) \mapsto x^4 + y^4 + u^4 + v^4 - 1$. Show that 0 is a regular value, and thus, S is a 3-dimensional topological manifold.

For the second part, cover S with 8 charts of the form

$$S_y^+ := \{(x, y, u, v) : y = \sqrt[4]{1 - x^4 - u^4 - v^4}\}, \quad S_y^- := \{(x, y, u, v) : y = -\sqrt[4]{1 - x^4 - u^4 - v^4}\}.$$

Then, show that the transition functions are C^1.

8. Assume that an orientable regular surface S is minimal, meaning that the mean curvature is constant 0, however, S has no planar points.

- Show that the Gauss map $N : S \to S^2$ has the following property:

$$\langle dN_P(w_1), dN_P(w_2)\rangle = -K(P)\langle w_1, w_2\rangle$$

  for all $P \in S$ and all $w_1, w_2 \in T_P(S)$ where $K(P)$ is the Gaussian curvature at the point P.

- Show that the above condition implies that the angle of two smooth curves $\alpha : \mathbb{R} \to S$ and $\beta : \mathbb{R} \to S$ intersecting at a point P on S is the same up to a sign as the angle of their images under the Gauss map ($N \circ \alpha$ and $N \circ \beta$) at the point $N(P)$. (In this part, you can assume that $N \circ \alpha$ and $N \circ \beta$ are smooth curves.)

Recall: the angle at P of two smooth curves intersecting at a point P is the angle between the tangent vectors of the curves at the point P.

Sketch of a solution: For the first part, realize that it is enough to show that the property holds for the eigenvectors of dN_P. Luckily, the eigenvalues satisfy $\lambda_1 = -\lambda_2$ because the surface is minimal, moreover

$$\lambda_1^2 = \lambda_2^2 = -K(P).$$

For the second part, realize that the angle γ between two vectors v and w satisfies

$$\cos\gamma = \frac{v \cdot w}{||v|| \cdot ||w||} = \frac{\langle v, w\rangle}{\sqrt{\langle v, v\rangle \cdot \langle w, w\rangle}}.$$

Thus, if all $\langle ., . \rangle$ gets multiplied by the same number, the absolute value of $\cos\gamma$ is unchanged implying the statement.
