Markov Chains and Applications

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Written examination: Thursday 12 January 2023, 9:00–12:00.

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation. Formulas alone are not enough. Formulate your answers clearly and carefully.
- Open book exam: lecture notes, textbooks and handwritten notes can be consulted.
- To each question is associated a score written in boldface, e.g. 9 points are associated to question (1a). *Total score:* 100.

Exercise 1

Consider a sequence of independent rolls of a fair die (i.e. a die with 6 equiprobable faces). Let $(X_n)_{n\geq 1}$ denote the minimum of the outcome of the first n rolls.

(1a) **[9 pts]** Argue that $(X_n)_{n\geq 1}$ is a Markov chain and write down its transition matrix.

(1b) **[3 pts]** Is this chain irreducible?

(1b) **[5 pts]** For an arbitrary $n \in \mathbb{N}$, compute the probability that $X_n > 2$.

Exercise 2

Fix some $M \in \mathbb{N}_0$. Consider the discrete-time Markov chain with state space $S = \{0, 1, 2, \dots, M\}$ and transition probabilities as follows: for any $i \in S$, $p_{i,i+1} = 1 - \frac{i}{M}$, $p_{i,i-1} = \frac{i}{M}$, and $p_{i,j} = 0$ otherwise.

(2) **[9 pts]** Find a probability measure π on S that satisfies the *detailed balance condition* with respect to the transition probabilities specified above.

HINT: you may want to use the classical Newton's binomial formula $(a+b)^N = \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k}$*, valid for any real numbers* a, b *and* $N \in \mathbb{N}$.

Exercise 3

Consider the Simple Symmetric Random Walk (SSRW) on the integer lattice starting from the origin (that is, the Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ with state space $\mathbb{Z}$ such that $Z_0 = 0$, and with transition probabilities as follows: $p_{x,y} = 1/2$ if $y \in \{x+1, x-1\}$ and $p_{x,y} = 0$ otherwise).

(3) **[13 pts]** Fix $n \in \mathbb{N}_0$. By means of the reflection principle, compute the probability that in the first $3n$ steps, the SSRW is at the origin only at times $0, n$ and $3n$. In other words, you are required to compute

$$P_0\big(Z_n = 0, Z_{3n} = 0, Z_k \neq 0 \quad \forall k \in \{1,2,3,\dots,3n-2,3n-1\} \setminus \{n\}\big).$$

Exercise 4

(4a) **[8 pts]** Describe what the *Kendall symbolic representation* of a queing system is, explaining in particular the meaning of the five different symbols.

(4b) **[5 pts]** Explain in words what the *PASTA principle* is and why it is valid in case of an $M/M/s$ queue model.

Exercise 5

Given a finite connected unoriented graph $G = (V, E)$ with n vertices (i.e. $|V| = n$), we call *simple continuous-time random walk* the continuous-time markov chain with state space V jumping at rate 1 by choosing uniformly a neighbouring vertex. A *system of n-coalescening random walks* on G is a collection of n simple continuous-time random walkers starting from the n distinct vertices of the graph, and which move independently until two such walks meet, in which case, they merge together and they continue travelling as a single independent simple continuous-time random walk.

(5a) **[9 pts]** Describe and draw a graphical representation for a system of n-coalescening random walks on G.

(5b) **[5 pts]** Give the definition and describe the generator of the so-called *voter model* on G (with two opinions $\{0,1\}$).

(5c) **[8 pts]** Consider the voter model $(\eta_t)_{t\geq 0}$ on G starting from an arbitrary configuration $\eta_0 \in \{0,1\}^V$ and let

$$\tau_{cons} := \inf\{t \geq 0 : \eta_t \in \{\underline{0}, \underline{1}\}\}$$

be the associated *consesus time* (i.e. , the first time the voter model reaches one of its absorbing states). Argue via the graphical representation that τ_{cons} in the voter model is bounded from above by the so-called *coalescing time* τ_{coal} associated to a system of n-coalescening random walks. The latter τ_{coal} being defined as the first time that all the n random walks in the system merge.

HINT: note that the Poisson arrows in the space-time graphical representation of the voter model have the same distribution when seen backward or forward in time. Thus, the same randomness encoded in the field of Poisson arrows can be used to compare the voter model and n-coalescing random walkers.

(5d) **[8 pts]** For an edge $e = (x, y) \in E$, we say that this edge is *discordant* at time $T > 0$ for the voter model $(\eta_t)_{t\geq 0}$ on G if $\eta_T(x) \neq \eta_T(y)$. Express the probability that a given edge $e \in E$ is discordant at time $T > 0$ for the voter model in terms of the starting configuration η_0, and of an event involving two independent random walks starting from x and y.

Exercise 6

Consider the subshift of finite type X_A with $S = \{0, 1, 2, 3\}$ and

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(6a) **[4 pts]** Recall that L_n, $n \geq 1$, denotes the number of allowed words of length n in X_A. Compute L_3.

(6b) **[4 pts]** Denote by $P(x)$ the polynomial $P(x) = x^4 - x^3 - x^2 - x - 1$. You can assume without proof that $P(x)$ has two real roots $\lambda_1 > 1$ and $\lambda_2 \in (-1, 0)$ and two complex roots $\lambda_3, \overline{\lambda}_3 \in \mathbb{C}$ with $|\lambda_3| < 1$. Prove that $h(X_A) = \log \lambda_1$.

(6c) **[4 pts]** Write $C = 1/(2\lambda_1^4 - \lambda_1^3 + \lambda_1 + 2)$. Show that the vector $u = (\lambda_1^3, \lambda_1^2, \lambda_1, 1)^\top$ is a right eigenvector of A with eigenvalue λ_1 and $w = C(\lambda_1, \lambda_1^2 - \lambda_1, 1 + 1/\lambda_1, 1)$ is a left eigenvector of A with eigenvalue λ_1. Also show that $w \cdot u = 1$.

(6d) **[6 pts]** Compute the Parry measure in terms of λ_1 and C.