Markov Chains and Applications

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Written examination: Monday 30 January 2023, 9:00–12:00.

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation. Formulas alone are not enough. Formulate your answers clearly and carefully.
- Open book exam: lecture notes, textbooks and handwritten notes can be consulted.
- To each question is associated a score written in boldface, e.g. 3 points are associated to question (1a). *Total score:* 100.

Exercise 1

Fix $M \in \mathbb{N}$ and $\epsilon \in (0, 1/2)$, and let $(X_n)_{n\geq 1}$ be the Markov chain with state space $S := \{0, 1, 2, \ldots, M-1, M\}$ and the following tranisition matrix P :

$$P(0,0) = \epsilon, \quad P(0,i) = \frac{1-\epsilon}{M}, \text{ for all } i \in S \setminus \{0\},$$

$$\text{and } \ P(i,j) = \frac{1}{M}, \text{ for any } \ i,j \in S \setminus \{0\}.$$

(1a) **[3 pts]** Is this chain irreducible?

(1b) **[3 pts]** For any $i \in S$, show that $\lim_{n\to\infty} P^n(i,0) = 0$.

(1c) **[6 pts]** For any $i \in S$ and $j \in S \setminus \{0\}$, show that $\lim_{n\to\infty} P^n(i,j) = \frac{1}{M}$.
(*HINT: you may want to use Chapman-Kolmogorov equations.)*

(1d) **[3 pts]** Determine all the probability measures on S that are invariant for P.

Exercise 2

Fix $L \in \mathbb{N}$ and let $(X_n)_{n\geq 1}$ be the Markov chain with state space $S := \{0, 1, 2, \ldots, L-1, L\}$ and tranisition matrix P defined as follows:

$$P(0,0) = P(0,1) = \frac{1}{2}, \quad P(L,L) = 1,$$

$$\text{and } \ P(i,i+1) = P(i,i-1) = \frac{1}{2}, \text{ for all } \ 1 \leq i \leq L-1.$$

(2a) **[7 pts]** Define τ_L to be the first time that the chain is in state L and argue that for all $i \in S$:

$$P(\tau_L \leq n | X_0 = i) \geq P(\sigma \leq n/L),$$

where σ is a random variable with values in $\mathbb{N}$ and Geometric distribution of parameter 2^{-L}, that is :

$$P(\sigma = n) = \left(1 - 2^{-L}\right)^{n-1} 2^{-L}, \quad \forall n \in \mathbb{N}$$

(2b) **[7 pts]** Show that for any $i \in S$, $\lim_{n\to\infty} P^n(i, L) = 1$.

(2c) **[3 pts]** Determine all the probability measures on S that are invariant for P.

Exercise 3

Let P be the transition matrix of an irreducible and aperiodic Markov chain on a finite state space S.

(3a)**[3 pts]** Consider a stochastic process $(Z_n)_{n\geq 1}$ with values in $S \times S$ defined as follows. For any $n \geq 1$, $Z_n := \left(X_n^{(1)}, X_n^{(2)}\right)$, where $\left(X_n^{(1)}\right)_{n\geq 1}$, $\left(X_n^{(2)}\right)_{n\geq 1}$ are two independent copies of the Markov chain with transition matrix P. Show that $(Z_n)_{n\geq 1}$ is also a Markov chain and write down the entries of its transition matrix.

(3b)**[10 pts]** For a given $s \in S$, set $\tau := \min\{n \geq 0 : Z_n = (s, s)\}$ to be the first time Z_n is in state (s, s), and assume that $P(\tau < \infty | Z_0 = (i, j)) = 1$, for any $i, j \in S$. Show (without invoking any known Theorem about Markov chains) that

$$\lim_{n\to\infty} |P^n(i, l) - P^n(j, l)| = 0, \quad \text{for any } i, j, l \in S.$$

(3c)**[5 pts]** Consider a probability measure π on S which is invariant for P. Show (without invoking any known Theorem about Markov chains) that

$$\lim_{n\to\infty} P^n(i, j) = \pi(j) \quad \text{for any } i, j \in S.$$

Exercise 4

(4a) **[15 pts]** Let $G = (V, E)$ be an arbitrary undirected connected graph with $n = |V|$ vertices and let $f : V \to \mathbb{R}$ be a real valued function defined on the set of vertices of the graph.

Define an appropriate Markov chain on S and describe in details a related Monte Carlo algorithm to find a vertex for which f is maximal or approximately so.

Exercise 5

Let $Q = (q_{i,j})_{i,j \in S}$ be the infinitesimal generator of a continuous-time Markov chain $(X_t)_{t \geq 0}$ on a finite state space S.

(5a) **[6 pts]** Define a transition matrix P of a discrete time Markov chain on S and an algorithm based on P to sample a trajectory of $(X_t)_{t \geq 0}$ in a given time interval $[0, T]$, for $T > 0$ fixed.

(5b) **[9 pts]** Assume that Q is such that $q_{i,i} \neq q_{j,j}$ for some $i \neq j \in S$. As in the previous question, define a transition matrix P' (but P' must be different from the matrix P in your previous answer) of a discrete time Markov chain on S and an algorithm based on P' to sample a trajectory of $(X_t)_{t \geq 0}$ in a given time interval $[0, T]$, for $T > 0$ fixed.

(5c) **[2 pts]** Which of the two algorithms you described in the previous answers is more convenient in practice? Motivate your answer.

Exercise 6

Consider the subshift of finite type with $S = \{0, 1, 2\}$ and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(6a) **[5 pts]** Show that this subshift of finite type is irreducible.

(6b) **[5 pts]** Prove that $h(X_A) = \log \lambda$, where λ is the largest root of the polynomial $P(x) = x^3 - x^2 - 1$.

(6c) **[8 pts]** Show that $(1/\lambda, \lambda, 1)$ is a left eigenvector of A with eigenvalue λ and compute the Parry measure in terms of λ.