ANALYTIC NUMBER THEORY, 26/1/2017

(1) a) $\sum_{m\geq 1}^{N} a_m g(m) = \sum_{m\geq 1}^{N} (A(m) - A(m-1))\, g(m)$ $(A(0) := 0)$

$= \sum_{m\geq 1}^{N} A(m) g(m) - \sum_{m=2}^{N} A(m-1)\, g(m) = \sum_{m\geq 1}^{N-1} A(m)\,(g(m) - g(m+1)) + A(N) g(N)$

$= A(N) g(N) - \sum_{m\geq 1}^{N-1} A(m) \int_m^{m+1} g'(t)\,dt = A(N) g(N) - \int_1^N A(t)\, g'(t)\,dt.$

b) By a) we have for $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma$, taking $A(x) := \sum_{n\leq x} f(n)$,

$(*)\ \sum_{m\geq 1}^{N} f(n) n^{-s} = A(N) N^{-s} - \int_1^N A(t) \cdot (-s t^{-s-1})\,dt = A(N) N^{-s} + s \int_1^N A(t) t^{-s-1}\,dt$

Note that $|A(N) \cdot N^{-s}| \leq C \cdot N^{\sigma} \cdot N^{-\operatorname{Re} s} \to 0$ as $N \to \infty$

$|A(t) t^{-s-1}| \leq C \cdot t^{\sigma} \cdot t^{-\operatorname{Re} s - 1} = C t^{\sigma - \operatorname{Re} s - 1}$ and $\int_1^{\infty} C t^{\sigma - \operatorname{Re} s - 1}\,dt$ converges.

Hence the right-hand side of $(*)$ converges and so $L_f(s)$ converges for $\operatorname{Re} s > \sigma$.

c) b) implies that $L_\mu(s) = \sum_{n\geq 1}^{\infty} \mu(n) n^{-s}$ has abscissa of convergence $\leq \frac{1}{2} + \varepsilon$ for every $\varepsilon > 0$, so it has abscissa of convergence $\leq \frac{1}{2}$. We use the fact from the lecture notes that $L_\mu(s) = \zeta(s)^{-1}$ for $s \in \mathbb{C}$, $\operatorname{Re} s > 1$. By b), $L_\mu(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \operatorname{Re} s > \frac{1}{2}\}$. Hence $\zeta(s)^{-1}$ has an analytic continuation to $\{s \in \mathbb{C} : \operatorname{Re} s > \frac{1}{2}\}$. This implies $\zeta(s) \neq 0$ for $\operatorname{Re} s > \frac{1}{2}$.

2 a) Theorem Let f be an arithmetic function such that
(i) $f(n) \geq 0$ for all n;
(ii) there $C \geq 0, \sigma \geq 0$ such that $|\sum_{n \leq x} f(n)| \leq C x^\sigma$ for all x.
(iii) $L_f(s)$ converges for $s \in \mathbb{C}$, $\operatorname{Re} s > \sigma$;
(iv) There is $\alpha \in \mathbb{C}$ such that $g(s) := L_f(s) - \frac{\alpha}{s-\sigma}$ has an analytic continuation to an open set containing $\{s \in \mathbb{C} : \operatorname{Re} s \geq \sigma\}$
Then $\lim\limits_{x \to \infty} \frac{1}{x^\sigma} \sum\limits_{n \leq x} f(n)$ exists and is equal to $\frac{\alpha}{\sigma}$.

b) Recall that $\chi(n), \mu(n)$ assume their values in $\{-1, 0, 1\}$.
We want to apply our Tauberian theorem to $f(n) = \mu(n)\chi(n) + 1$.
We show that it satisfies (i)–(iv) with $\sigma = \alpha = 1$.
(i) is clear since $\mu(n)\chi(n) \in \{-1, 0, 1\}$ for all n.
(ii) $|\sum\limits_{n \leq x} f(n)| = |\sum\limits_{n \leq x} (\mu(n)\chi(n) + 1)| \leq 2x$

(iii) follows from (ii) and exercise 1 b).
(iv) Consider $L_f(s) = \sum_{n \geq 1}^\infty (\mu(n)\chi(n) + 1) n^{-s} = \sum_{n \geq 1}^\infty (\mu(n)\chi(n)) n^{-s} + \zeta(s)$

Note that $\sum_{n \geq 1}^\infty \mu(n)\chi(n) n^{-s} = \prod\limits_p (1 + \sum\limits_{k \geq 1}^\infty \mu(p^k)\chi(p^k) p^{-ks})$

$= \prod\limits_p (1 - \chi(p) p^{-s})$ for $s \in \mathbb{C}$, $\operatorname{Re} s > 1$, since $\mu\chi$ is multiplicative and $\sum_{n \geq 1}^\infty \mu(n)\chi(n) n^{-s}$ converges absolutely for $\operatorname{Re} s > 1$.
On the other hand, $L(s, \chi) = \prod_p (1 - \chi(p) p^{-s})^{-1}$ for $\operatorname{Re} s > 1$.
So $L_{\mu\chi}(s) = L(s, \chi)^{-1}$ for $\operatorname{Re} s > 1$
Hence $L_f(s) = L(s, \chi)^{-1} + \zeta(s)$ for $\operatorname{Re} s > 1$.
Recall that $L(s, \chi)$ is analytic on $\{\operatorname{Re} s > 0\}$ and $L(s, \chi) \neq 0$ if $\operatorname{Re} s = 1$. Further, $\zeta(s)$ is analytic on $\{\operatorname{Re} s > 0\} \setminus \{1\}$ and has residue 1 at $s = 1$; so $\zeta(s) - \frac{1}{s-1}$ has an analytic continuation to $\{\operatorname{Re} s > 0\}$. It follows that $L_f(s) - \frac{1}{s-1}$ has an analytic continuation to an open set containing $\{s \in \mathbb{C} : \operatorname{Re} s \geq 1\}$
It follows that $\lim\limits_{x \to \infty} \frac{1}{x} \sum\limits_{n \leq x} f(n) = \lim\limits_{x \to \infty} \frac{1}{x} \sum\limits_{n \leq x} (\mu(n)\chi(n) + 1) = 1$,
hence $\lim\limits_{x \to \infty} \frac{1}{x} \sum\limits_{n \leq x} \mu(n)\chi(n) = 0$.

3) a) By exercise 1 b), $L_f(s)$ has abscissa of convergence ≤ 0, so it is analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$. Further, $\zeta(s)$ is analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\} \setminus \{1\}$ with a simple pole at $s=1$. If $L_f(1) = 0$, this pole is cancelled by the zero of $L_f(s)$. Hence $\zeta(s) L_f(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$

b) We use that if $L_f(s), L_g(s)$ converge absolutely for $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma$, then so does $L_{f*g}(s)$ and $L_{f*g}(s) = L_f(s) L_g(s)$. Moreover, if f, g are multiplicative arithmetic functions, then so is $f*g$.
Now $\zeta(s)$, $L_f(s)$ (with f as in a)) converge absolutely for $\operatorname{Re} s > 1$, and $\zeta(s) L_f(s) = L_{E*f}(s)$ for $\operatorname{Re} s > 1$, where $E(n) = 1$ for all n. Further E, f are multiplicative, hence so is $g := E*f$.
We have

$$g(p^k) = \sum_{j=0}^{k} E(p^{k-j}) f(p^j) = \sum_{j=0}^{k} f(p)^j = \begin{cases} k+1 & \text{if } f(p) = 1 \\ 1 & \text{if } f(p) = 0 \\ 1 & \text{if } f(p) = -1,\ k \text{ even} \\ 0 & \text{if } f(p) = -1,\ k \text{ odd.} \end{cases}$$

c) We use the following theorem. If h is an arithmetic function with $h(n) \geq 0$ for all n and such that $L_h(s)$ has abscissa of convergence σ, then $L_h(s)$ does not have an analytic continuation to any open set containing σ.
We apply this with the function g from b). By a), $L_g(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$. By b) we have $g(n) \geq 0$ for all n, and moreover, $g(p^k) \geq 1$ if k is even so $g(n) \geq 1$ if n is a square. Hence

$$L_g(\tfrac{1}{2}) = \sum_{n \geq 1}^{\infty} g(n) n^{-\frac{1}{2}} \geq \sum_{m \geq 1}^{\infty} g(m^2) m^{-1} \geq \sum_{m \geq 1}^{\infty} m^{-1} = \infty,$$

i.e., $L_g(s)$ has abscissa of convergence $\geq \frac{1}{2}$. This contradicts the Theorem, so $L_f(1)$ must be $\neq 0$

(4)

(4) a) We use that $\int_0^1 e(\alpha t)\,d\alpha = \begin{cases} 1 & \text{if } t=0 \\ 0 & \text{if } t\neq 0 \end{cases}$

Note that $Q(\alpha)^2 C(\alpha) e(-\alpha n)$

$$= \Big(\sum_{\substack{x_1\in\mathbb{Z}\\ 1\le x_1\le n^{1/2}}} e(\alpha x_1^2)\Big)\Big(\sum_{\substack{x_2\in\mathbb{Z}\\ 1\le x_2\le n^{1/2}}} e(\alpha x_2^2)\Big)\Big(\sum_{\substack{y\in\mathbb{Z}\\ 1\le y\le n^{1/3}}} e(\alpha y^3)\Big)\big(e(-\alpha n)\big)$$

$$= \sum_{\substack{1\le x_1, x_2\le n^{1/2}\\ 1\le y\le n^{1/3}}} e(\alpha(x_1^2+x_2^2+y^3-n))$$

So $\int_0^1 Q(\alpha)^2 C(\alpha) e(-\alpha n)\,d\alpha = \int_0^1 \sum(\ldots)\,d\alpha = \sum \int_0^1 e(\alpha(x_1^2+x_2^2+y^3-n))\,d\alpha$

$$= T(n).$$

b) $\Big|\sum_{\substack{m\in\mathbb{Z}\\ 1\le m\le b}} e(\theta m)\Big| = \Big| e(\theta)\frac{e(\theta b)-1}{e(\theta)-1}\Big|$ (sum formula for geometric progressions)

$$\le \frac{1+|e(\theta b)|}{|e(\theta)-1|} \le \frac{2}{|e(\frac{\theta}{2}) - e(\frac{-\theta}{2})|} = \frac{1}{|\sin(\pi\theta)|}$$

$$\le \frac{\pi/2}{\pi\theta} = \frac{1}{2\theta} \quad \text{using } \sin x \ge \frac{2}{\pi}x \text{ for } 0\le x\le \frac{\pi}{2}.$$

c) $|K(q)|^2 = K(q)\overline{K(q)} = \sum_{\substack{m_1\in\mathbb{Z}\\ 1\le m_1\le q}} e\Big(\frac{m_1^2+m_1+7}{q}\Big) \sum_{\substack{m_2\in\mathbb{Z}\\ 1\le m_2\le q}} e\Big(-\frac{m_2^2+m_2+7}{q}\Big)$

$$= \sum_{\substack{m_1, m_2\in\mathbb{Z}\\ 1\le m_1, m_2\le q}} e\Big(\frac{m_1^2+m_1+7-m_2^2-m_2-7}{q}\Big)$$

~~Write m_1~~ Define $h_1\in\{1,\ldots,q\}$ by $m_1\equiv m_2+h_1 \pmod q$. Note that this gives a 1-1 correspondence $(m_1, m_2)\leftrightarrow(h_1, m_2)$ with $m_1, m_2, h_1\in\{1,\ldots,q\}$

So

$$|K(q)|^2 = \sum_{\substack{h_1, m_2 \in \mathbb{Z} \\ 1 \le h_1, m_2 \le q}} e\left(\frac{m_2^2 + 2h_1 m_2 + h_1^2 + m_2 + h_1 + 7 - m_2^2 - m_2 - 7}{q}\right)$$

$$= \sum_{\substack{h_1, m_2 \in \mathbb{Z} \\ 1 \le h_1, m_2 \le q}} e\left(\frac{2h_1 m_2 + h_1^2 + h_1}{q}\right) = \sum_{h_1 = 1}^{q} e\left(\frac{h_1^2 + h_1}{q}\right) \cdot \sum_{m_2 = 1}^{q} e\left(\frac{2h_1 m_2}{q}\right)$$

If $h_1 = q$, then $\sum_{m_2=1}^{q} e\left(\frac{2q m_2}{q}\right) = q$, otherwise, this sum is equal to

$$e\left(\frac{2h_1}{q}\right) \cdot \frac{e\left(\frac{2h_1 q}{q}\right) - 1}{e\left(\frac{2h_1}{q}\right) - 1} = 0.$$

So $|K(q)|^2 = e\left(\frac{q^2 + q}{q}\right) \cdot q = q.$