Final Exam "Analyse 4"

Monday, July 4, 10.00 – 13.00

- Write your name and student ID number on every page.
- Clear your table completely leaving only a pen and a non-graphical calculator.
- This exam has *five* problems. Do not forget the problems on the back.

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1. *(14 points)* Given is a function $v : \mathbb{R}^2 \to \mathbb{R}$ with

$$v(x, y) = 2y^3 - 6x^2 y + \frac{1}{2}(y^2 - x^2)\,.$$

(a) Show that v is harmonic.

(b) Find a function $u : \mathbb{R}^2 \to \mathbb{R}$ such that the complex function

$$f(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic. Is u unique? Motivate your answer.

(c) The function f from (b) is given as a function of x and y. Write it as a function of $z = x + iy$.

2. *(23 points)* Let $a \in \mathbb{C}$ be such that $|a| < 1$ and let n be a natural number with $n \geq 1$. Furthermore, define the function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = (z - 1)^n e^z - a$$

and let $H = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$.

(a) Show that for any zero z of f in H it holds true that $|z - 1| < 1$.

(b) How many zeros (counted with multiplicity) does f have in H.

(c) How many *different* zeros does f have in H.

3. *(17 points)*

(a) State and prove Liouville's theorem on bounded, entire functions.

(b) Let $f, g \colon \mathbb{C} \to \mathbb{C}$ be holomorphic functions and assume that there is $M \geq 0$ such that $|f(z)| \leq M|g(z)|$ for all $z \in \mathbb{C}$. Assume further that g has no zeros in $\mathbb{C}$. Show that there is a constant C such that $f(z) = Cg(z)$ for all $z \in \mathbb{C}$.

(c) Let $f, g \colon \mathbb{C} \to \mathbb{C}$ be holomorphic functions and assume that there is $M \geq 0$ such that $|f(z)| \leq M|g(z)|$ for all $z \in \mathbb{C}$. Show that there is a constant C such that $f(z) = Cg(z)$ for all $z \in \mathbb{C}$.

4. *(28 points)*

(a) Consider

$$f(z) = \frac{(e^{iz} - 1)(1 - \cos(2z))}{z^4 \sinh(z)}$$

on its natural domain of definition in the complex plane.
(*Hint*: Recall that $\sinh(z) = \frac{1}{2}(\exp(z) - \exp(-z))$.)

i. Determine all singularities of f and their type, that is, distinguish between removable singularities, poles or essential singularities. For poles, also specify their order.

ii. Determine the principal part of the Laurent series around $z = 0$.

(b) Let $n \in \mathbb{N}$ and $g_n : \mathbb{C} \setminus \{\pm i\} \to \mathbb{C}, z \mapsto (z^2 + 1)^{-n}$.

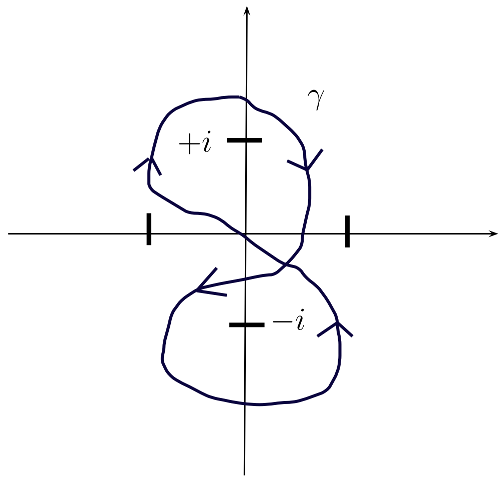
i. Show that

$$\operatorname{Res}(g_n, \pm i) = \mp i \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} \frac{1}{2^{2n-1}} .$$

ii. Determine the value of the complex line integral

$$\oint_\gamma g_n(z)\, dz ,$$

where the curve γ is given below.



5. *(18 points)* Use residue calculus to compute the value of the definite integral

$$\int_0^\infty \frac{1}{(x^2+4)^2} \cos(\mu x)\, dx \quad (\mu \in \mathbb{R}) .$$

(*Hint*: You may use that $\lim_{R\to\infty} \int_0^\pi \exp(\alpha R \sin(t))\, dt = 0 , \alpha < 0$.)