Final Exam “Analyse 4”
Monday, July 4, 10.00 – 13.00

- Write your name and student ID number on every page.
- Clear your table completely leaving only a pen and a non-graphical calculator.
- This exam has five problems. Do not forget the problems on the back.

1. (14 points) Given is a function \(v : \mathbb{R}^2 \to \mathbb{R}\) with
\[
v(x, y) = 2y^3 - 6x^2y + \frac{1}{2}(y^2 - x^2).
\]
(a) Show that \(v\) is harmonic.
(b) Find a function \(u : \mathbb{R}^2 \to \mathbb{R}\) such that the complex function
\[
f(x + iy) = u(x, y) + iv(x, y)
\]
is holomorphic. Is \(u\) unique? Motivate your answer.
(c) The function \(f\) from (b) is given as a function of \(x\) and \(y\). Write it as a function of \(z = x + iy\).

2. (23 points) Let \(a \in \mathbb{C}\) be such that \(|a| < 1\) and let \(n\) be a natural number with \(n \geq 1\). Furthermore, define the function \(f : \mathbb{C} \to \mathbb{C}\) by
\[
f(z) = (z - 1)^ne^z - a
\]
and let \(H = \{z \in \mathbb{C} : \Re z \geq 0\}\).
(a) Show that for any zero \(z\) of \(f\) in \(H\) it holds true that \(|z - 1| < 1\).
(b) How many zeros (counted with multiplicity) does \(f\) have in \(H\).
(c) How many different zeros does \(f\) have in \(H\).

3. (17 points)
(a) State and prove Liouville’s theorem on bounded, entire functions.
(b) Let \(f, g : \mathbb{C} \to \mathbb{C}\) be holomorphic functions and assume that there is \(M \geq 0\) such that \(|f(z)| \leq M|g(z)|\) for all \(z \in \mathbb{C}\). Assume further that \(g\) has no zeros in \(\mathbb{C}\). Show that there is a constant \(C\) such that \(f(z) = Cg(z)\) for all \(z \in \mathbb{C}\).
(c) Let \(f, g : \mathbb{C} \to \mathbb{C}\) be holomorphic functions and assume that there is \(M \geq 0\) such that \(|f(z)| \leq M|g(z)|\) for all \(z \in \mathbb{C}\). Show that there is a constant \(C\) such that \(f(z) = Cg(z)\) for all \(z \in \mathbb{C}\).
4. (28 points)

(a) Consider

\[ f(z) = \frac{(e^{iz} - 1)(1 - \cos(2z))}{z^4 \sinh(z)} \]

on its natural domain of definition in the complex plane.

(Hint: Recall that \( \sinh(z) = \frac{1}{2}(\exp(z) - \exp(-z)) \).)

i. Determine all singularities of \( f \) and their type, that is, distinguish between removable singularities, poles or essential singularities. For poles, also specify their order.

ii. Determine the principal part of the Laurent series around \( z = 0 \).

(b) Let \( n \in \mathbb{N} \) and \( g_n : \mathbb{C} \setminus \{\pm i\} \to \mathbb{C}, z \mapsto (z^2 + 1)^{-n} \).

i. Show that 

\[ \text{Res}(g_n, \pm i) = \mp i \left( \frac{2n - 2}{n - 1} \right) \frac{1}{2^{2n-1}}. \]

ii. Determine the value of the complex line integral

\[ \oint_{\gamma} g_n(z) \, dz, \]

where the curve \( \gamma \) is given below.

5. (18 points) Use residue calculus to compute the value of the definite integral

\[ \int_{0}^{\infty} \frac{1}{(x^2 + 4)^2} \cos(\mu x) \, dx \quad (\mu \in \mathbb{R}). \]

(Hint: You may use that \( \lim_{R \to \infty} \int_{0}^{\pi} \exp(\alpha R \sin(t)) \, dt = 0, \alpha < 0 \).)