1. **(14 points)** Given is a function $u : \mathbb{R}^2 \to \mathbb{R}$ with

$$u(x, y) = xy - 2x^3 + 6xy^2.\)**

(a) Show that $u$ is harmonic.

(b) Find a function $v : \mathbb{R}^2 \to \mathbb{R}$ such that the complex function

$$f(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic. Is $v$ unique? Motivate your answer.

(c) The function $f$ from (b) is given as a function of $x$ and $y$. Write it as a function of $z = x + iy$.

2. **(20 points)** Let $U = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$ be the complex plane slit along the negative real axis and let $\text{Log} : U \to \mathbb{C}$ be the principal branch of the complex logarithm. Consider the analytic function $h : U \to \mathbb{C}$ given by

$$h(z) = \text{Log}(z) - 4(z - 2)^2.$$

(a) Show that $h$ has precisely two zeros (counted with multiplicity) on $U_1(2) = \{z \in \mathbb{C} : |z - 2| < 1\}$.

(b) Show that $h$ has two different zeros on $U_1(2)$.

3. **(22 points)** Consider the function $f : \mathbb{C} \setminus \{\pm 1\} \to \mathbb{C}$ given by

$$f(z) = \frac{3z - 1}{z^2 - 1}.$$

(a) Determine the radius of convergence of $f$ around $-4 - 4i$.

(b) Determine the Laurent series of $f$ on the open annulus

$$\{z \in \mathbb{C} \mid 1 < |z + 2| < 3\}.$$
4. (30 points) Consider the function

\[ f(z) = \frac{z^3}{1 + z^4} \exp(-iz) \]

on its natural domain of definition in the complex plane.

(a) Determine all singularities of \( f \) and their type, that is, distinguish between removable singularities, poles or essential singularities. For poles, also specify their order.

(b) Compute for all singularities of \( f \) in the lower half plane their residues and show that their sum is

\[ \frac{1}{2} \exp\left(-\frac{1}{2} \sqrt{2}\right) \cos\left(\frac{1}{2} \sqrt{2}\right). \]

(c) Determine the value of the complex line integral

\[ \oint_{\gamma} f(z) \, dz, \]

where the curve \( \gamma \) is given below.

(d) Determine the value of the real definite integral

\[ \int_{0}^{\infty} \frac{x^3}{1 + x^4} \sin(-x) \, dx. \]

(Hint: You may use that \( \lim_{R \to \infty} \int_{0}^{\pi} \exp(-R \sin(t)) \, dt = 0. \))

5. (14 points)

(a) Give a precise explanation why there cannot be an analytic function \( f : \mathbb{C} \to \mathbb{C} \) such that

\[ f(1/n) = \frac{1}{1 + (1/n^2)} \]

for all \( n \in \mathbb{N} \).

(b) Assume that for an analytic function \( g : \mathbb{C} \to \mathbb{C} \) it holds true that \( \text{Re}(g(z)) \leq 2016 \) for all \( z \in \mathbb{C} \). Show that \( g \) must be constant.

Note: Part (a) and (b) can be solved independently of each other.