Final Exam "Analyse 4"

Thursday, June 16, 10.00 – 13.00

- Write your name and student ID number on every page.
- Clear your table completely leaving only a pen and a non-graphical calculator.
- This exam has *five* problems. Do not forget the problems on the back.

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1. *(14 points)* Given is a function $u : \mathbb{R}^2 \to \mathbb{R}$ with

$$u(x, y) = xy - 2x^3 + 6xy^2 \,.$$

(a) Show that u is harmonic.

(b) Find a function $v : \mathbb{R}^2 \to \mathbb{R}$ such that the complex function

$$f(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic. Is v unique? Motivate your answer.

(c) The function f from (b) is given as a function of x and y. Write it as a function of $z = x + iy$.

2. *(20 points)* Let $U = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$ be the complex plane slit along the negative real axis and let $\mathrm{Log} : U \to \mathbb{C}$ be the principal branch of the complex logarithm. Consider the analytic function $h : U \to \mathbb{C}$ given by

$$h(z) = \mathrm{Log}(z) - 4(z - 2)^2 \,.$$

(a) Show that h has precisely two zeros (counted with multiplicity) on $U_1(2) = \{z \in \mathbb{C} : |z - 2| < 1\}$.

(b) Show that h has two *different* zeros on $U_1(2)$.

3. *(22 points)* Consider the function $f : \mathbb{C} \setminus \{\pm 1\} \to \mathbb{C}$ given by

$$f(z) = \frac{3z - 1}{z^2 - 1} \,.$$

(a) Determine the radius of convergence of f around $-4 - 4i$.

(b) Determine the Laurent series of f on the open annulus

$$\{z \in \mathbb{C} \mid 1 < |z + 2| < 3\} \,.$$

4. *(30 points)* Consider the function

$$f(z) = \frac{z^3}{1+z^4}\,\exp(-iz)$$

on its natural domain of definition in the complex plane.

(a) Determine all singularities of f and their type, that is, distinguish between removable singularities, poles or essential singularities. For poles, also specify their order.

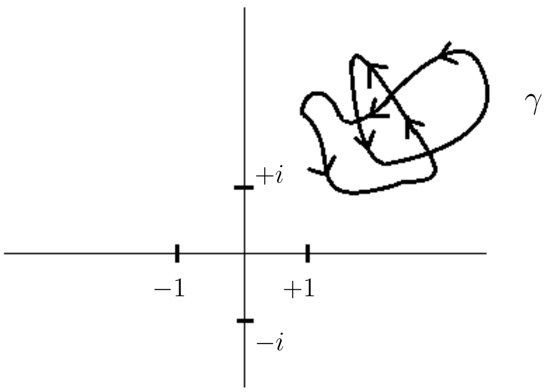
(b) Compute for all singularities of f in the lower half plane their residues and show that their sum is

$$\frac{1}{2}\exp\left(-\frac{1}{2}\sqrt{2}\right)\cos\left(\frac{1}{2}\sqrt{2}\right).$$

(c) Determine the value of the complex line integral

$$\oint_\gamma f(z)\,dz\,,$$

where the curve γ is given below.



(d) Determine the value of the real definite integral

$$\int_0^\infty \frac{x^3}{1+x^4}\,\sin(-x)\,dx\,.$$

(*Hint*: You may use that $\lim_{R\to\infty}\int_0^\pi \exp(-R\sin(t))\,dt = 0$.)

5. *(14 points)*

(a) Give a precise explanation why there cannot be an analytic function $f:\mathbb{C}\to\mathbb{C}$ such that

$$f(1/n) = \frac{1}{1+(1/n^2)}$$

for all $n\in\mathbb{N}$.

(b) Assume that for an analytic function $g:\mathbb{C}\to\mathbb{C}$ it holds true that $\mathrm{Re}(g(z))\le 2016$ for all $z\in\mathbb{C}$. Show that g must be constant.

Note: Part (a) and (b) can be solved independently of each other.