Exam: Modular Forms	**7 June 2016**
Mastermath	**10:00–13:00**

Note:

- The use of books, lecture notes, calculators, etc. is *not* allowed.
- If you are unable to answer a subitem, you are still allowed to use the result in the remainder of the exercise.
- Throughout this exam, N and k denote positive integers.

1. This question tests your general knowledge of the material. If you use terminology or notation from the course in answering a subquestion, you should define these.
   (a) Define what a *modular form* of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$ is.
   (b) Show that if k is odd, then the only weakly modular function of weight k for the group $\Gamma_1(2)$ is the zero function.
   (c) Let f be a meromorphic function on $\mathbb{H}$ that is not identically zero, is weakly modular of weight k for $\mathrm{SL}_2(\mathbb{Z})$, and is meromorphic at the cusp ∞ of $\mathrm{SL}_2(\mathbb{Z})$. State (without proof) the *valence formula* relating the weight of f to the orders of the zeroes and poles of f in a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.
   (d) Let $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Define the *diamond operator* $\langle d \rangle$ on $\mathrm{M}_k(\Gamma_1(N))$, and show that the result is independent of any choices made in the definition.
   (e) Define the subspace of *oldforms* (also called the *old subspace*) $\mathrm{S}_k(\Gamma_1(N))_{\mathrm{old}}$ inside the space $\mathrm{S}_k(\Gamma_1(N))$.
   (f) Let E be an elliptic curve over $\mathbb{Q}$. Recall that the conjecture of Birch and Swinnerton-Dyer predicts that the (algebraic) rank of E equals the order of vanishing of the L-function $L(E,s)$ at $s=1$. Explain how modular forms are used to show that $L(E,s)$ is well defined around $s=1$.

2. (a) Let Γ be a congruence subgroup. Show that the number of cusps of Γ is less than or equal to the index of Γ in $\mathrm{SL}_2(\mathbb{Z})$.
   (b) Let p be a prime number. Show that the congruence subgroup $\Gamma_0(p)$ has exactly two cusps, namely 0 and ∞.
   (c) Show that $\frac{1}{2}$ is an irregular cusp of $\Gamma_1(4)$.

3. In this question, you may use without proof that a basis for the space $\mathrm{S}_3(\Gamma_1(9))$, expressed as q-expansions at the cusp ∞, is given by (f_1, f_2) with
$$\begin{aligned} f_1 &= q \qquad - 6q^3 + q^4 + 6q^5 + 9q^6 + O(q^7),\\ f_2 &= \quad q^2 - 3q^3 + q^4 + 2q^5 + 3q^6 + O(q^7). \end{aligned}$$
   (a) Prove that the matrix of the Hecke operator T_3 on $\mathrm{S}_3(\Gamma_1(9))$, with respect to the basis (f_1, f_2), equals $\left(\begin{smallmatrix} -6 & -3 \\ 9 & 3 \end{smallmatrix}\right)$.
   (b) Show that the operator T_3 on $\mathrm{S}_3(\Gamma_1(9))$ has two linearly independent normalised eigenforms g_1 and g_2, and compute the first six terms of the q-expansions of g_1 and g_2.

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4. Let Γ be a congruence subgroup, and let f and g be in $\mathrm{M}_k(\Gamma)$.

   (a) Prove that the function $z \mapsto f(z)\overline{g(z)}(\Im z)^k$ (where $\Im z$ denotes the imaginary part of z) is Γ-invariant.

   (b) Suppose that at least one of f and g is in $\mathrm{S}_k(\Gamma)$. Define the *Petersson inner product* $\langle f, g\rangle_\Gamma$ of f and g, taking convergence issues into account, and show that the result is independent of any choices made in the definition.

   (c) Define the *Eisenstein subspace* $\mathcal{E}_k(\Gamma)$ of $\mathrm{M}_k(\Gamma)$.

   (d) Show that the dimension of the space $\mathcal{E}_k(\Gamma)$ is less than or equal to the number of cusps of Γ.

Maximum scores per subitem			
1a: 6	2a: 4	3a: 6	4a: 6
1b: 4	2b: 6	3b: 9	4b: 8
1c: 6	2c: 6		4c: 4
1d: 5			4d: 6
1e: 6			
1f: 8			
Maximum total = 90			
Mark = 1 + Total/10			