Mathematical Institute	Exam Time Series
Leiden University	June 25, 2014

Give concise, but rigorous arguments. Always motivate your answers. Use of calculators is neither allowed nor useful. Good luck!

1. (a) Formulate and prove the central limit theorem for the sample mean $\overline{X}_n = n^{-1}\sum_{t=1}^{n} X_t$, with X_t a linear process.

   (b) Let X_t and $\widetilde{X}_t$ be two solutions to the same GARCH equation. Formulate and prove a theorem that shows that asymptotically, as $t \to \infty$, the difference $X_t - \widetilde{X}_t$ becomes negligible.

2. (a) Let X_t be a moving average $X_t = Z_t - \theta Z_{t-1}$, where Z_t is a white noise series with variance σ^2 and we assume $|\theta| < 1$. Show that the best (linear) predictor of X_{t+1} given the infinite past $X_t, X_{t-1}, \ldots$ is $\widetilde{\Pi}_t X_{t+1} \equiv -\sum_{j=1}^{\infty} \theta^j X_{t+1-j}$ (try to be as formal as possible, but partial answers may also be acceptable).

   (b) Determine the square prediction error of the predictor $\widetilde{\Pi}_t X_{t+1}$ from (a).

3. (a) Consider the process Z_t as given and show that the bivariate process

$$\mathbf{X}_t = \begin{pmatrix} 1 & \theta \\ \theta & 0 \end{pmatrix} \begin{pmatrix} Z_t \\ Z_{t-1} \end{pmatrix}$$

   solves the recursive equation

$$\mathbf{X}_{t+1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{X_t} + \begin{pmatrix} 1 \\ \theta \end{pmatrix} Z_{t+1}.$$

   (b) Recall that a bivariate (real-valued) time series $\mathbf{X}_t$ is stationary, if its mean $\mathbb{E}[\mathbf{X}_t] = (\mathbb{E}[X_{t,1}], \mathbb{E}[X_{t,2}])^T$ and auto-covariance

$$\gamma_X(h) = \mathbb{E}[(\mathbf{X}_{t+h} - \mathbb{E}[\mathbf{X}_{t+h}])(\mathbf{X}_t - \mathbb{E}[\mathbf{X}_t])^T]$$

   do not dependend on t. Let Z_t be a white noise with variance σ^2. Show that the process $\mathbf{X}_t$ from (a) is stationary.

   (c) Show that the process X_t in (a) is the unique stationary solution to the recursive equation in (a). *Hint*: consider the difference between two solutions.

   (d) Let Z_t be a white noise and let $Y_t = Z_t + \theta Z_{t-1}$ be a moving average. Take Y_t as an output process, $\mathbf{X}_t$ from (a) as a state process and show that Y_t can be written in the (linear) state space form.

Continued on the other side!

4. (a) Let $X_t = Z_t + \theta Z_{t-1}$ be a moving average (here Z_t is a white noise with variance σ^2). Does the time series $Y_t = \frac{1}{2}(X_t + X_{t-1})$ have a spectral density? If yes, provide an expression for it.

5. (a) What does causality of an ARMA process mean? What does invertibility of an ARMA process mean?
   (b) Show that there exists a stationary solution to the following equation,

$$X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t,$$

   where Z_t is an i.i.d. Gaussian white noise. Is this solution also causal and invertible?

6. (a) Provide a brief argument (e.g. relying on a result from the lecture notes) why the periodogram $I_n(\lambda)$ is an inconsistent estimator of the spectral density $f_X(\lambda)$.

7. Assume the time series X_t is a stationary auto-regressive process of order 1 and with unknown parameters φ and σ^2. Suppose you observe $X_1, \ldots, X_n$.
   (a) Find the least squares estimator of φ. Next use this to find an estimator of σ^2.
   (b) Show that in the present setting the least squares estimator of parameter φ is consistent (try to be as formal as possible, but partial answers may also be acceptable).

Norming

1(a): 2; **1(b)**: 2.
2(a): 2; **2(b)**: 2.
3(a): 1; **3(b)**: 2; **3(c)**: 2; **3(d)**: 2.
4(a): 2.
5(a): 1; **5(b)**: 2.
6(a): 1.
7(a): 2; **7(b)**: 2.

The final grade depends on the results of the homework assignments as well.