Probability: Coupling Theory

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Written examination: Wednesday 8 June 2016, 14:00–17:00.

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation. Formulas alone are not enough. Formulate your answers clearly and carefully.
- The use of textbooks, lecture notes or handwritten notes is not allowed.
- The questions below are weighted as follows: (1) 3, 10; (2) 10, 2, 5; (3) 5, 5; (4) 5, 10; (5) 5, 5; (6) 10, 10; (7) 5, 10. *Total:* 100. Pass: ≥ 55; no pass: ≤ 54.

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(1) (a) Consider two random variables X and X' with laws $\mathbb{P}$ and $\mathbb{P}'$ on the same measurable space $(E, \mathcal{E})$.

Give the definition of a coupling of these two random variables.

(b) Consider two sequences of random variables $X = (X_n)_{n\in\mathbb{N}_0}$ and $X' = (X'_n)_{n\in\mathbb{N}_0}$ taking values in $(E^{\mathbb{N}_0}, \mathcal{E}^{\otimes\mathbb{N}_0})$, where $(E, \mathcal{E})$ is a given measurable space. Let $(\hat{X}, \hat{X}')$ be a coupling of X and X' with law $\hat{\mathbb{P}}$ and let $T := \inf\{n \in \mathbb{N}_0 \mid \hat{X}_m = \hat{X}'_m \text{ for all } m \geq n\}$ be the so–called *coupling time.*

Show that, for every $n \geq 0$, the total variation distance between the laws of X_n and X'_n is bounded from above by $2\hat{\mathbb{P}}(T > n)$.

(2) (a) Let $S = (S_n)_{n\in\mathbb{N}_0}$ be the simple symmetric random walk on $\mathbb{Z}$ starting from the origin and denote by $\mathbb{P}$ its law. Show that, for every $k \in \mathbb{Z}$ even, the following holds:

$$\lim_{n\to\infty} \|\mathbb{P}(S_n \in \cdot) - \mathbb{P}(S_n + k \in \cdot)\|_{TV} = 0, \tag{1}$$

with $\|\cdot\|_{TV}$ denoting *total variation distance.*

(b) What can you say about the left hand side of equation (1) in case $k \in \mathbb{Z}$ is odd?

(c) Can the result in equation (1) be generalized to dimensions $d \geq 2$? Motivate your answer.

(3) (a) Give the definition of a *random card shuffle* and describe three different examples.

(b) Give the definition of *a sequence of threshold times* for random card shuffles and explain its interpretation.

(4) (a) Let $X = (X_n)_{n\in\mathbb{N}}$ be a sequence of independent $\{0,1\}$–valued Bernoulli random variables with $\mathbb{P}(X_n = 1) := p_n \in (0,1), n \geq 1$. For a given big N, set $S_N = \sum_{n=1}^{N} X_n$ and explain in which sense S_N can be approximated by a Poisson random variable.

(b) State and prove one such Poisson approximation for S_N.

(5) (a) Let μ be a probability measure on $\mathbb{R}$. Given two real–valued non–decreasing functions f and g, show that μ satisfies the so–called FKG inequality, that is:

$$\mu[fg] \geq \mu[f]\mu[g], \tag{2}$$

where $\mu[f]$ denotes the expectation of f with respect to μ.

(b) Consider now a finite set S and let μ be a probability measure on the power set $\mathcal{P}(S)$ associated to S. Is it possible to state the FKG inequality as in equation (2) in this setting? Explain what a *monotone* real–valued function $f : \mathcal{P}(S) \longrightarrow \mathbb{R}$ means in this setting, and which assumption on μ suffices.

(6) (a) Define the so–called *directed site percolation* on $\mathbb{Z}^2$, the associated random subset called *cluster of the origin*, and the so–called *critical percolation threshold* denoted by p_c.

(b) Sketch the main lines of the proof of the following upper bound :

$$p_c \leq 80/81.$$

(7) (a) Give the definition of the *contact process* with intensity $\lambda > 0$ on $\{0,1\}^{\mathbb{Z}^d}$.

(b) Denote by P^λ the semigroup of the *contact process* with intensity λ, and by $\delta_{[1]}$ the delta measure associated to the configuration $[1]$ with all 1's. Show that, for all $t \geq 0$, the following stochastic domination holds:

$$\delta_{[1]} P_t^{\lambda_1} \preceq \delta_{[1]} P_t^{\lambda_2},$$

with $0 < \lambda_1 < \lambda_2 < \infty$.