Examination for the course on

Random Walks

Teacher: L. Avena

Monday 09 January 2017, 10:00–13:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or diktaat is not allowed.
- There are 17 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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(1) **[10]** Consider the simple random walk $(S_n)_{n=0}^N$ of finite-length $N \in \mathbb{N}$ on the integers starting at 0 associated to the finite probability space (Ω_N, P_N). By means of *Kolmogorov extension theorem*, explain how the finite-length simple random walk can be uniquely extended to infinite-time horizon.

(2) Consider simple random walk $(S_n)_{n\in\mathbb{N}_0}$ on the d-dimensional integer lattice $\mathbb{Z}^d$. State the corresponding:

(a) **[5]** Strong law of large numbers and central limit theorem.

(b) **[5]** Large deviation principle for the position of the random walk when $d = 1$.

(3) **[5]** Compute the effective resistance of an infinite rooted regular tree with degree 3 (i.e. each node has 3 children) when the edges have unit resistance.

(4) Consider a finite connected directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ under the assumption that $xy \in \mathcal{E} \iff yx \in \mathcal{E}$. To each edge $xy \in \mathcal{E}$ associate a symmetric *conductance* $C_{xy} = C_{yx} \in (0, \infty)$ and set $C_x = \sum_{y\neq x} C_{xy}$. Consider the Markov chain on the vertex set $\mathcal{V}$ with transition matrix $P = (P_{xy})_{x,y\in\mathcal{V}}$ where $P_{xy} = C_{xy}/C_x$. Fix two distinct points $a, b \in \mathcal{V}$.

(a) **[10]** Define the discrete Laplacian Δ associated to the Markov chain and what an harmonic function on $\mathcal{V} \setminus \{a, b\}$ with respect to Δ is. State further the related *Maximum* and *Uniqueness principles.*

(b) **[10]** For $x \in \mathcal{V}$, let p_x be the probability that the Markov chain starting from x hits a before b, that is, $p_x = P_x(\tau_a < \tau_b)$ where τ_a and τ_b denote the hitting times of a and b, respectively. By setting the appropriate Dirichlet problem, show that p_x can be interpreted in terms of a *voltage.*

(5) Let c_n denote the number of self-avoiding walks of length $n \in \mathbb{N}$ on the triangular lattice (i.e., the two-dimensional lattice where unit triangles are packed together).

(a) **[5]** What inequality is satisfied by $n \mapsto c_n$, and why does this inequality imply the existence of the so-called connective constant μ?

(b) **[5]** Compute c_3.

(c) **[5]** Show that $2^n \le c_n \le 6 \times 5^{n-1}$, $n \in \mathbb{N}$ and use this to obtain bounds on μ.

(6) (a) **[5]** Define the path space $\mathcal{W}_n$ of the pinned polymer of length $n \in \mathbb{N}$. Let $\bar{P}_n$ be the uniform measure on $\mathcal{W}_n$. The path measure with interaction strength $\zeta \in \mathbb{R}$ is

$$\bar{P}_n^\zeta(w) = \frac{1}{Z_n^\zeta} e^{\zeta \sum_{i=1}^n 1_{\{w_i=0\}}} \bar{P}_n(w), \qquad w \in \mathcal{W}_n,$$

Explain what this path measure models and the role of ζ.

(b) **[5]** Consider the function $\zeta \mapsto f_n(\zeta) = \frac{1}{n} \log Z_n^\zeta$. Compute and show how its first and second derivatives are related to the fraction of absorbed monomers (i.e. points of the path on the horizontal line).

(c) **[5]** Let $\zeta \mapsto f(\zeta)$ be the *free energy*, we saw that

$$f(\zeta) = \begin{cases} 0, & \text{if } \zeta \le 0, \\ \frac{1}{2}\zeta - \frac{1}{2}\log(2 - e^{-\zeta}), & \text{if } \zeta > 0. \end{cases} \tag{1}$$

Draw a plot of $f(\zeta)$ and, by using the fact that $\lim_{n\to\infty} f_n(\zeta) = f(\zeta)$, explain the *phase transition* of this model.

(d) **[Bonus]** Give a sketch of the proof of the existence and the non-negativity of the free energy. (HINT: you may want to use that $a(k)/b(k) \le Ck$, for all $k \in 2\mathbb{N}$ and some $C \in (0,\infty)$ where $a(k) = P(\sigma > k)$, $b(k) = P(\sigma = k)$, and σ stands for the first return time to 0 of the simple random walk.)

(7) (a) **[5]** Give a definition of the one-dimensional Wiener process (also known as standard Brownian motion) $(W_t)_{t\ge 0}$.

(b) **[10]** State the one-dimensional heat equation with initial condition determined by a given bounded continuous function $f : \mathbb{R} \to \mathbb{R}$. Explain how this equation is related to the one-dimensional Wiener process.

(8) Consider the one period binomial asset pricing model. Suppose that the current price of a stock is $S_0 = 80$ euro, and that at the end of the period of time its price must be either $S_1 = 40$ or $S_1 = 160$ euro. A European call option on the stock is available with a striking price of $K = 100$ euro, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest.

(a) **[5]** Compute the arbitrage-free price of this call option.

(b) **[5]** Give the replicating portfolio for this option and explain what this means.