Examination for the course:

Random Walks

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Monday 25 January 2016, 14:00–17:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or diktaat is not allowed.
- There are 10 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

(1) Consider simple random walk \((S_n)_{n=0}^{N}\) of finite-length \(N \in \mathbb{N}\) on the integers starting at 0.

(a) [5] Given two stopping times \(T_1\) and \(T_2\), is the minimum \(T = \min\{T_1, T_2\}\) again a stopping time? Prove your answer!

(b) [5] Define \(T_3 = \max\{0 \leq k \leq N : S_k = 0\}\). Is \(T_3\) a stopping time? Prove your answer!

(2) Consider simple random walk \((S_n)_{n \in \mathbb{N}_0}\) on the integer lattice \(\mathbb{Z}\).

(a) [5] State the Large Deviation Principle (LDP) for the position of the random walk and describe its interpretation.

(b) [10] Give the main lines of the proof of the LDP. (HINT: For \(a \in (0, 1]\), use that the maximum of \(\binom{n}{k}\) with \(k \geq (1 + a)n/2\) is attained at \([((1 + a)n/2)]\) and use that \(\lim_{n \to \infty} \frac{1}{n} \log Q_n(a) = -\frac{1 + a}{2} \log \frac{1 + a}{2} - \frac{1 - a}{2} \log \frac{1 - a}{2}\), where \(Q_n(a) = \left(\binom{n}{((1 + a)n/2)}\right)^n\).)

(3) [5] Compute the effective resistance between \(a\) and \(b\) of the following network of unit resistances:

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 a
 ▼
 ▼
 ▼
 ▼
 ▼
 b
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(4) Consider a finite connected directed graph \(G = (\mathcal{V}, \mathcal{E})\) under the assumption that \(xy \in \mathcal{E} \iff yx \in \mathcal{E}\). To each edge \(xy \in \mathcal{E}\) associate a symmetric conductance \(C_{xy} = C_{yx} \in (0, \infty)\) and set \(C_x = \sum_{y \sim x} C_{xy}\). Consider the Markov chain on the vertex set \(\mathcal{V}\) with transition matrix \(P = (P_{xy})_{x,y \in \mathcal{V}}\) where \(P_{xy} = C_{xy}/C_x\). Fix two distinct points \(a, b \in \mathcal{V}\).
(a) Define the discrete Laplacian $\Delta$ associated to the Markov chain and define what an harmonic function on $V \setminus \{a, b\}$ with respect to $\Delta$ is. State further the related Maximum and Uniqueness principles.

(b) For $x \in V$, let $p_x$ be the probability that the Markov chain starting from $x$ hits $a$ before $b$, that is, $p_x = P_x(\tau_a < \tau_b)$ where $\tau_a$ and $\tau_b$ denote the hitting times of $a$ and $b$, respectively. By formulating the proper Dirichlet problem, show that $p_x$ can be interpreted in terms of a voltage.

(5) Let $c_n$ denote the number of self-avoiding walks of length $n \in \mathbb{N}$ on the triangular lattice (i.e., the two-dimensional lattice where unit triangles are packed together).

(a) What inequality is satisfied by $n \mapsto c_n$, and why does this inequality imply the existence of the so-called connective constant $\mu$?

(b) Compute $c_3$.

(c) Show that $2^n \leq c_n \leq 6 \times 5^{n-1}$, $n \in \mathbb{N}$ and use this to obtain bounds on $\mu$.

(6) Give the definition of the path space $W_n$ of the pinned polymer of length $n \in \mathbb{N}$. The path measure with interaction strength $\zeta \in \mathbb{R}$ is

$$\tilde{P}_n^\zeta(w) = \frac{1}{Z_n} e^{\zeta \sum_{i=1}^n 1(w_i = 0)} \tilde{P}_n(w), \quad w \in W_n.$$ 

Explain what this path measure models.

(b) Consider the function $\zeta \mapsto f_n(\zeta) = \frac{1}{n} \log Z_n^\zeta$. Compute and show how its first derivative is related to the fraction of absorbed monomers (i.e., points of the path on the horizontal line).

(c) Let $\zeta \mapsto f(\zeta)$ be the free energy. We saw that

$$f(\zeta) = \begin{cases} 
0, & \text{if } \zeta \leq 0, \\
\frac{1}{2} \zeta - \frac{1}{2} \log(2 - e^{-\zeta}), & \text{if } \zeta > 0. 
\end{cases}$$

(1)

Draw a qualitative plot of $f'(\zeta)$ and, by using the fact that $\lim_{n \to \infty} f_n'(\zeta) = f'(\zeta)$, explain the phase transition of this model.

(d) Give a sketch of the proof of the existence and the non-negativity of the free energy. (HINT: you may want to use that $a(k)/b(k) \leq Ck$ for all $k \in 2\mathbb{N}$ and some $C \in (0, \infty)$ where $a(k) = P(\sigma > k)$, $b(k) = P(\sigma = k)$, and $\sigma$ stands for the first return time to 0 of the simple random walk.)

(7) (a) Give a definition of the one-dimensional Wiener process (also known as standard Brownian motion) $(W_t)_{t \geq 0}$.

(b) Let $(W_t)_{t \geq 0}$ be a standard Brownian motion. Put $X_t = \sqrt{2}(W_t - W_t/2)$. Is $(X_t)_{t \geq 0}$ a standard Brownian motion? Prove your answer.

(8) Suppose that the current price of a stock is $S_0 = 100$ euro, and that at the end of a period of time its price must be either $S_1 = 75$ or $S_1 = 150$ euro. A call option on the stock is available with a striking price of $K = 90$ euro, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest.

(a) Compute the arbitrage-free price of the call option.

(b) Suppose that you can buy such an option on the market for €40. What should you do? Explain your answer.