Examination for the course:

Random Walks

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Monday 25 January 2016, 14:00–17:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or diktaat is not allowed.
- There are 10 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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(1) Consider simple random walk $(S_n)_{n=0}^N$ of finite-length $N \in \mathbb{N}$ on the integers starting at 0.

(a) **[5]** Given two stopping times T_1 and T_2, is the minimum $T = \min\{T_1, T_2\}$ again a stopping time? Prove your answer!

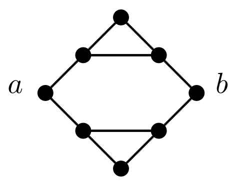
(b) **[5]** Define $T_3 = \max\{0 \leq k \leq N\colon S_k = 0\}$. Is T_3 a stopping time? Prove your answer!

(2) Consider simple random walk $(S_n)_{n\in\mathbb{N}_0}$ on the integer lattice $\mathbb{Z}$.

(a) **[5]** State the Large Deviation Principle (LDP) for the position of the random walk and describe its interpretation.

(b) **[10]** Give the main lines of the proof of the LDP. (*HINT*: For $a \in (0,1]$, use that the maximum of $\binom{n}{k}$ with $k \geq (1+a)n/2$ is attained at $\lceil (1+a)n/2 \rceil$ and use that $\lim_{n\to\infty} \frac{1}{n} \log Q_n(a) = -\frac{1+a}{2} \log \frac{1+a}{2} - \frac{1-a}{2} \log \frac{1-a}{2}$, where $Q_n(a) = \binom{n}{\lceil (1+a)n/2 \rceil}$.)

(3) **[5]** Compute the effective resistance between a and b of the following network of unit resistances:



(4) Consider a finite connected directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ under the assumption that $xy \in \mathcal{E} \iff yx \in \mathcal{E}$. To each edge $xy \in \mathcal{E}$ associate a symmetric *conductance* $C_{xy} = C_{yx} \in (0, \infty)$ and set $C_x = \sum_{y\sim x} C_{xy}$. Consider the Markov chain on the vertex set $\mathcal{V}$ with transition matrix $P = (P_{xy})_{x,y\in\mathcal{V}}$ where $P_{xy} = C_{xy}/C_x$. Fix two distinct points $a, b \in \mathcal{V}$.

(a) **[5]** Define the discrete Laplacian Δ associated to the Markov chain and define what an harmonic function on $\mathcal{V} \setminus \{a, b\}$ with respect to Δ is. State further the related *Maximum* and *Uniqness principles*.

(b) **[10]** For $x \in \mathcal{V}$, let p_x be the probability that the Markov chain starting from x hits a before b, that is, $p_x = P_x(\tau_a < \tau_b)$ where τ_a and τ_b denote the hitting times of a and b, respectively. By formulating the proper Dirichlet problem, show that p_x can be interpreted in terms of a *voltage*.

(5) Let c_n denote the number of self-avoiding walks of length $n \in \mathbb{N}$ on the triangular lattice (i.e., the two-dimensional lattice where unit triangles are packed together).

(a) **[5]** What inequality is satisfied by $n \mapsto c_n$, and why does this inequality imply the existence of the so-called connective constant μ?

(b) **[5]** Compute c_3.

(c) **[5]** Show that $2^n \leq c_n \leq 6 \times 5^{n-1}$, $n \in \mathbb{N}$ and use this to obtain bounds on μ.

(6) (a) **[5]** Give the definition of the path space $\mathcal{W}_n$ of the pinned polymer of length $n \in \mathbb{N}$. The path measure with interaction strength $\zeta \in \mathbb{R}$ is

$$\bar{P}_n^{\zeta}(w) = \frac{1}{Z_n^{\zeta}} e^{\zeta \sum_{i=1}^{n} 1_{\{w_i=0\}}} \bar{P}_n(w), \qquad w \in \mathcal{W}_n.$$

Explain what this path measure models.

(b) **[5]** Consider the function $\zeta \mapsto f_n(\zeta) = \frac{1}{n} \log Z_n^{\zeta}$. Compute and show how its first derivative is related to the fraction of *absorbed monomers* (i.e., points of the path on the horizontal line).

(c) **[5]** Let $\zeta \mapsto f(\zeta)$ be the *free energy*. We saw that

$$f(\zeta) = \begin{cases} 0, & \text{if } \zeta \leq 0, \\ \frac{1}{2}\zeta - \frac{1}{2}\log(2 - e^{-\zeta}), & \text{if } \zeta > 0. \end{cases} \tag{1}$$

Draw a qualitative plot of $f'(\zeta)$ and, by using the fact that $\lim_{n\to\infty} f_n'(\zeta) = f'(\zeta)$, explain the *phase transition* of this model.

(d) **[Bonus]** Give a sketch of the proof of the existence and the non-negativity of the free energy. (*HINT*: you may want to use that $a(k)/b(k) \leq Ck$ for all $k \in 2\mathbb{N}$ and some $C \in (0, \infty)$ where $a(k) = P(\sigma > k)$, $b(k) = P(\sigma = k)$, and σ stands for the first return time to 0 of the simple random walk.)

(7) (a) **[5]** Give a definition of the one-dimensional Wiener process (also known as standard Brownian motion) $(W_t)_{t\geq 0}$.

(b) **[10]** Let $(W_t)_{t\geq 0}$ be a standard Brownian motion. Put $X_t = \sqrt{2}(W_t - W_{t/2})$. Is $(X_t)_{t\geq 0}$ a standard Brownian motion? Prove your answer.

(8) Suppose that the current price of a stock is $S_0 = 100$ euro, and that at the end of a period of time its price must be either $S_1 = 75$ or $S_1 = 150$ euro. A call option on the stock is available with a striking price of $K = 90$ euro, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest.

(a) **[5]** Compute the arbitrage-free price of the call option.

(b) **[5]** Suppose that you can buy such an option on the market for €40. What should you do? Explain your answer.