Re-examination for the course on

Random Walks

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Tuesday 8 July 2014, 10:00–13:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with an explanation.
- The use of notes and/or diktaat is not allowed.
- There are 10 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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(1) **[5]** Consider simple random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ with length $N \in \mathbb{N}$ starting at 0. Which of the following two times is a stopping time? Prove your answer!

$$T_1 = \lfloor \tfrac{1}{2} N \rfloor, \qquad T_2 = \max\{0 \leq k \leq N\colon\ S_k = 0\}.$$

(2) **[5]** Consider a random walk on the square lattice $\mathbb{Z}^2$ with "diagonal jumps", i.e., the jump probabilities are:

$$P(X_1 = x) = \begin{cases} \frac{1}{4}, & \text{if } x \in \{(0,1),(1,1),(0,-1),(-1,-1)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Compute the covariance matrix $(\mathrm{Cov}(X_1^{(i)}, X_1^{(j)})_{i,j=1,2}$, where $X_1^{(i)}$ is the i-th component of X_1, and state the central limit theorem for the partial sums $S_n = \sum_{i=1}^n X_1$, $n \in \mathbb{N}$.

(3) **[10]** Use the reflection principle to compute the probability $P(S_m \leq 2 \text{ for } 0 \leq m \leq 5)$ for a simple random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$. In other words: What is the probability that a simple random walk does not exceed the value 2 during its first 5 steps?

(4) **[5]** Compute the effective resistance between 0 and 2 of the graph with vertex set $\{0,1,2\}$ and edge set as follows: 2 edges between vertices 0 and 1; 4 edges between vertices 1 and 2; 1 edge between vertices 0 and 2.

(5) Given is a finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with conductances $C_{xy} \in (0,\infty)$ assigned to each of the edges $xy \in \mathcal{E}$ such that $C_{xy} = C_{yx}$. Pick any two vertices $a, b \in V$ and place a battery across them.

(a) **[5]** Formulate the Dirichlet Principle.

(b) **[5]** Formulate the Thomson Principle.

(c) **[10]** Express the effective resistance of $\mathcal{G}$ in terms of both these variational principles.

(6) Let c_n denote the number of self-avoiding walks of length $n \in \mathbb{N}$ on the rooted tree of degree 6 starting at the root.

(a) **[5]** Give a formula for c_n.

(b) **[5]** Show that c_n is at least as large as the number of self-avoiding walks of length n on $\mathbb{Z}^3$.

(c) **[5]** What is c_n on the unrooted tree?

(7) Consider the wetted polymer (= pinned polymer with hard wall constraint) of length $n \in \mathbb{N}$ with interaction strength $\zeta \in \mathbb{R}$.

(a) **[5]** Give the definition of the free energy $\zeta \mapsto f^+(\zeta)$.

(b) **[10]** Explain why $f^+(\zeta) = f(\zeta - \zeta_c^+)$ with $\zeta_c^+ = \log 2$, where $f(\zeta)$ is the free energy of the pinned polymer.

(c) **[Bonus]** Explain how this formula is derived.

(8) **[10]** Let $(W_t)_{t\geq 0}$ be a standard Brownian motion. Put $X_t = W_{2t} - W_t$. Is $(X_t)_{t\geq 0}$ a Brownian motion?

(9) **[5]** Let $(W(t))_{t\geq 0}$ be standard Brownian motion. The Ornstein-Uhlenbeck process $(U_t)_{t\geq 0}$ is defined by

$$U(t) = e^{-t}W(e^{2t}).$$

Provide an explicit expression for $\text{cov}(U(s), U(t))$ for $s, t \geq 0$.

(10) Suppose that the current price of a stock is $S_0 = 50$ euro, and that at the end of a period of time its price must be either $S_1 = 25$ or $S_1 = 100$ euro. A call option on the stock is available with a striking price of $K = 50$ euro, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest.

(a) **[5]** Compute the arbitrage-free price of the call option.

(b) **[5]** Suppose that you can buy such an option on the market for €25. What should you do?

SOLUTIONS

(1) T_1 is a stopping time because it is deterministic. T_2 is not a stopping time. Take, for example, the situation where $S_1 = 1$. Then it is unclear whether this is already a stopping time or the walk returns to 1 before time N, and hence $\{T_1 = 1\} \not\subseteq \mathcal{A}_1$.

(2) We compute $\Sigma_{i,j} = \mathrm{Cov}(X_1^{(i)}, X_1^{(j)})$ $(i, j = 1, 2)$ term by term:

$$
\begin{aligned}
\Sigma_{1,1} &= \sum_{x\in\mathbb{Z}d} x^{(1)}\, x^{(1)}\, P(X = x) = \frac{1}{4}(1+1+1+1) = 1,, \\
\Sigma_{2,2} &= \sum_{x\in\mathbb{Z}d} x^{(2)}\, x^{(2)}\, P(X = x) = \frac{1}{2} \\
\Sigma_{1,2} &= \Sigma_{2,1} = \sum_{x\in\mathbb{Z}d} x^{(2)}\, x^{(1)}\, P(X = x) = \frac{1}{2}.
\end{aligned}
$$

Since $E(X_1) = 0$, the central limit theorem states that $n^{-1/2}\sum_{k=1}^{n} X_k$ converges in distribution to a $\mathcal{N}(0, \Sigma)$ distribution, with Σ as computed above. Note that the correlations of the individual jumps carry over to the scaling limit.

(3) See Eq. (1.47) of the lecture notes (reflection principle) and note that $P(S_5 = k) = 0$ when k is even, to get

$$
P(S_m \le 2 \text{ for } 0 \le m \le 5) = 1 - P(\sigma_3 \le 5) = 1 - \left[\sum_{k=3}^{5} P(S_5 = k) + \sum_{k=4}^{5} P(S_5 = k)\right]
$$
$$
= 1 - [P(S_5 = 3) + 2P(S_5 = 5)] = 1 - \tfrac{1}{2^5}\left[\binom{5}{3} + 2\binom{5}{5}\right] = 1 - \tfrac{1}{32}\,[5 + 2] = \tfrac{25}{32}.
$$

(4) The 2 edges with unit resistances between vertices 0 and 1 may be replaced by a single edge with resistance $\frac{1}{2}$. The 4 edges with unit resistances between vertices 1 and 2 may be replaced by a single edge with resistance $\frac{1}{4}$. The two single edges thus obtained may be replaced by a single edge between vertices 0 and 2 with resistance $\frac{3}{4}$. Together with the edge of unit resistance between vertices 0 and 2 that was already present, this gives an effective resistance of $[(\frac{3}{4})^{-1} + (1)^{-1}]^{-1} = \frac{3}{7}$.

(5) (a) The Dirichlet principle is stated in the diktaat and says that the total energy dissipation is the infimum over all unit potentials of the Dirichlet form.

(b) The Thomson principle is stated in the diktaat and says that the total energy dissipation is the infimum over all unit flows of the Dirichlet form.

(c) $1/E(v) = R_{\text{eff}} = \hat{E}(i)$, where v is the unit potential of the system (battery has $v_a = 1$, $v_b = 0$), i is the unit flow of the system (battery has $i_a = 1$, $i_b = -1$). The notations E and $\hat{E}$ are used to distinguish between the two cases.

(6) (a) $c_n = 5^n$ because each step has 5 possibilities. No loops are possible on a tree.

(b) The answer in (a) is an upper bound for the number of self-avoiding walks on the cubic lattice, because this lattice has 6 nearest-neighbours (like the tree with degree 6) but allows for loops. Proof via embedding.

(c) $c_n = 6 \times 5^{n-1}$ because the first step has 6 possibilities rather than 5.

(7) (a) $f^+(\zeta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n^{\zeta,+}$, where $Z_n^{\zeta,+}$ is the normalising partition sum in the definition of the path measure. The latter is the same as that for the pinned polymer with the constraint that no path may move below the interface.

(b) Each time the path hits the interface it loses the option to step downwards. Therefore the reward e^ζ that is picked at such a hit comes with a loss by a factor $\frac{1}{2}$ in the number of allowed paths. Hence the effective interaction strength ζ' for the wetted polymer satisfies the relation $e^{\zeta'} = e^\zeta \frac{1}{2}$, i.e., $\zeta' = \zeta - \zeta_c$ with $\zeta_c = \log 2$.

(c) See the diktaat.

(8) If $X_t = W_{2t} W_t$. then

$$X_{2t} - X_t = W_{4t} - W_{2t} - (W_{2t} - W_t) = W_{4t} - 2W_{2t} + W_t$$

and

$$\text{cov}(X_{2t} - X_t, X_t - X_0) = -t \neq 0.$$

Hence the increments are not independent, and so $(X_t)_{t\geq 0}$ is not a Brownian motion. Alternatively, note that

$$\begin{aligned} E[X_t X_s] &= E[W_{2t}W_{2s} - W_t W_{2s} - W_{2t} W_s + W_s W_t] \\ &= \min(2t, 2s) - \min(t, 2s) - \min(2t, s) + \min(t, s) \neq \min(t, s). \end{aligned}$$

(9) Since $EU(t) = 0$, we have

$$\text{cov}(U(s), U(t)) = e^{-s-t}\text{cov}(W(e^{2s}), W(e^{2t})) = e^{-s-t} \min(e^{2s}, e^{2t}) = e^{-|s-t|}.$$

(10) (a) We have $S_0 = 50$, $S_1(H) = 100 = uS_0$, $S_1(H) = 25 = dS_0$. Hence $u = 2$, $d = \frac{1}{2}$, and

$$q = \frac{1 + r - d}{u - d} = \frac{1.25 - 0.5}{2 - 0.5} = \frac{0.75}{1.5} = \frac{1}{2}.$$

Since the pay-off of a call option is $C(\omega) = (S_1(\omega) - 50)^+$, we get

$$C_0 = \frac{1}{1 + 0.25}\Big(\frac{1}{2} \times 50 + \frac{1}{2} \times 0\Big) = \frac{50}{2.5} = 20.$$

(b) The number of shares in the replicating portfolio is

$$\Delta_0 = \frac{C_1(H) - C_1(T)}{S_1(H) - S_1(T)} = \frac{50}{75} = \frac{2}{3}.$$

Hence, with the initial wealth of 20, we can replicate the option by buying two thirds of a share for $\frac{2}{3} \times 50 = \frac{100}{3} = 33\frac{1}{3}$ euro, and borrowing $33\frac{1}{3} - 20 = 13\frac{1}{3}$ euro on the market with interest $r = 25\%$.

If the option is sold for 25 euro, then we should proceed as follows:

- sell one option at 25;
- buy $\Delta_0 = \frac{2}{3}$ share of the stock, borrowing $\frac{100}{3} - 25 = \frac{25}{3} \approx 8.33$ euro on the market to do so.

At the end of the period we have settled the option, i.e., we have to pay either 50 when $\omega = H$ or 0 when $\omega = T$. We also have to pay back to the money market

$$\Big(\frac{100}{3} - 25\Big)\frac{5}{4} = \frac{125}{12} \approx 10.42.$$

Nevertheless, our position is covered since our debt (obligations), $50 + 10.42 = 60.42$ or 10.42, is lower than the value of two thirds of a share we own: $\frac{200}{3} \approx 66.67$ or $\frac{50}{3} \approx 16.67$. In both cases, we generate the same profit

$$\frac{200}{3} - 50 - \Big(\frac{100}{3} - 25\Big)\frac{5}{4} = \frac{50}{3} - \frac{125}{12} = \frac{75}{12} = 6.25,$$
$$\frac{50}{3} - 0 - \Big(\frac{100}{3} - 25\Big)\frac{5}{4} = \frac{50}{3} - \frac{125}{12} = \frac{75}{12} = 6.25.$$