Measure Theory 2014: Homework 3

Email your homework to Stein Bethuelsen,

bethuelsensa@math.leidenuniv.nl

or, place it in Stein's mailbox.

Homework is due on 22.10.2014, 11.15am (sharp!!!)

Final grade: Homework (90%)+Attendance (10%).

(1) [30 points] A measure ν on a σ-algebra $\mathcal{B}$ is called **complete** if for every $B \in \mathcal{B}$ such that $\nu(B) = 0$ and every $B' \subset B$, one has $B' \in \mathcal{B}$. In other words, every subset of a set of measure 0 belongs to the σ-algebra. Show that

(a) the measure $\mu_*|_{\tilde{\mathcal{A}}} = \mu^*|_{\tilde{\mathcal{A}}}$ on $\tilde{\mathcal{A}}$ from Exercise 3 of Homework 2,

(b) the Lebesgue measure λ^* on a σ-algebra $\mathcal{A}^*$ (Lebesgue measurable sets) constructed from the premeasure λ on the ring $\mathcal{F}_d$ (ring of figures in $\mathbb{R}^d$) using the Carathéodory Extenstion Theorem (Theorem 5.3, Bauer),

are **complete**.

Notation: $\mathfrak{B}(\mathbb{R}^d)$ is the Borel σ- algebra on $\mathbb{R}^d$, λ^d is the Lebesgue (–Borel) measure on $\mathfrak{B}(\mathbb{R}^d)$.

(2) /20pt/ Prove that $\mathfrak{B}(\mathbb{R})$ is also generated by the following collections of subsets of the real line:

(a) all intervals;

(b) all intervals with rational endpoints;

(c) all half-lines $[a, +\infty)$, $a \in \mathbb{R}$.

(3) /20pt/ True or false? Give a short justification, recalling appropriate theorems or providing examples:

- Any subset of a Borel-set is also a Borel-set.
- There is a subset A of $\mathbb{R}$ which is not Lebesgue measurable, but such that $B = \{x \in A : x \text{ is irrational}\}$ is Lebesgue measurable.

(4) /10pt/ Is the mid-third Cantor set (Homework 2) a Borel set?

(5) /20pt/ Part of Exercise 1b, Bauer, p.33: Prove that $B \in \mathfrak{B}(\mathbb{R}^d)$ has Lebesgue measure 0, if there is a covering of B by intervals $\{I_n : n \in \mathbb{N}\}$ such that

(i) $\sum_n \lambda^d(I_n) < \infty$;

(ii) every point $x \in B$ belongs to I_n for infinitely many n.

Hint: Use the result of Ex 1, Homework 1.