Probability: Coupling Theory

Teacher: L. Avena
Re-examination: Tuesday 28 June 2016, 14:00–17:00.

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation. Formulas alone are not enough. Formulate your answers clearly and carefully.
- The use of textbooks, lecture notes or handwritten notes is not allowed.
- The questions below are weighted as follows: (1) 5,5,5; (2) 10, 2, 5; (3) 5, 5; (4) 5, 10; (5) 5, 5; (6) 10, 10; (7) 5, 10. *Total:* 100. Pass: ≥ 55; no pass: ≤ 54.

---

(1) (a) Given two probability measures $\mathbb{P}$ and $\mathbb{P}'$ on the same measurable space $(E, \mathcal{E})$, their *total variation distance* is defined as

$$\|\mathbb{P} - \mathbb{P}'\|_{TV} := 2 \sup_{A \subset \mathcal{E}} [\,\mathbb{P}(A) - \mathbb{P}'(A)]. \tag{1}$$

Assume that the set E is *countable* and show that the definition in equation (1) can be rewritten as:

$$\|\mathbb{P} - \mathbb{P}'\|_{TV} = \sum_{x \in E} |\ \mathbb{P}(\{x\}) - \mathbb{P}'(\{x\})\ | \ .$$

(b) Give the definition of a coupling of two probability measures $\mathbb{P}$ and $\mathbb{P}'$ on the same measurable space $(E, \mathcal{E})$.

(c) Consider two sequences of random variables $X = (X_n)_{n \in \mathbb{N}_0}$ and $X' = (X'_n)_{n \in \mathbb{N}_0}$ taking values in $(E^{\mathbb{N}_0}, \mathcal{E}^{\otimes \mathbb{N}_0})$, where $(E, \mathcal{E})$ is a given measurable space. Let $(\hat{X}, \hat{X}')$ be a coupling of X and X' with law $\hat{\mathbb{P}}$ and let $T := \inf\{n \in \mathbb{N}_0 \mid \hat{X}_m = \hat{X}'_m \text{ for all } m \geq n\}$ be the so–called *coupling time*. Suppose that for some non-decreasing function $\phi : \mathbb{N}_0 \to [0, \infty)$ such that $\lim_{n\to\infty} \phi(n) = \infty$, we know that $\hat{\mathbb{E}}[\phi(T)] < \infty$. Under these assumptions, state and prove a convergence result for

$$\|\mathbb{P}(\theta^n X \in \cdot) - \mathbb{P}'(\theta^n X' \in \cdot)\|_{TV},$$

where θ denotes the *left–shift* operator on $E^{\mathbb{N}_0}$, i.e., $\theta(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$.

(2) (a) Let $S = (S_n)_{n\in\mathbb{N}_0}$ be the simple symmetric random walk on $\mathbb{Z}$ starting from the origin and denote by $\mathbb{P}$ its law. Show that, for every $k \in \mathbb{Z}$ even, the following holds:

$$\lim_{n\to\infty} \|\mathbb{P}(S_n \in \cdot) - \mathbb{P}(S_n + k \in \cdot)\|_{TV} = 0, \tag{2}$$

with $\|\cdot\|_{TV}$ denoting *total variation distance.*

(b) What can you say about the left hand side of equation (1) in case $k \in \mathbb{Z}$ is odd?

(c) Can the result in equation (1) be generalized to dimensions $d \geq 2$? Motivate your answer.

(3) (a) Define the notion of *strong uniform time* for a discrete-time Markov chain on a finite state space.

(b) Give the definition of the so-called *top-to-random shuffle* for a deck of N cards.

(c) Let T_N be the first time that the original *bottom card* is re-inserted in the deck while performing top-to-random shuffle. It is known that T_N is a strong uniform time. Let V_N denote the number of random draws with replacement from an urn with N balls until each ball has been drawn at least once. Show that T_N equals in distibution V_N and identify the distibution of V_N.

(d) It is known that the Markov chain associated to top-to-random shuffle admits a sequence of *threshold times.* Explain what this statement formally means and its heuristic interpration.

(e) Sketch the main lines of the proof of the statement in point ??(d) above.

(5) (a) Consider a finite set S and let μ_1 and μ_2 be two probability measures on the power set $\mathcal{P}(S)$ associated to S. After specifying the proper assumptions, state the so-called *Holley inequality.*

(b) In the context of point 5.(a), state and prove the so-called *FKG inequality.*

(a) (c) Assume S is the vertex set of a finite torus in $\mathbb{Z}^d$, identify $\mathcal{P}(S)$ with $\{0,1\}^S$ and let μ_β be the probability measure associated to the *Ising model* with given inverse temperature $\beta \in (0,\infty)$, that is:

$$\mu_\beta(a) = \frac{e^{\beta|\{x,y\in a: \|x-y\|=1\}|}}{Z_\beta}, \quad a \in \mathcal{P}(S),$$

with Z_β being a normalizing constant. Show that for arbitrary subsets $A, B \subset S$, the following correlation inequality holds:

$$\mu_\beta(\text{ all 1's on } A \cup B) \geq \mu_\beta(\text{ all 1's on } A)\mu_\beta(\text{ all 1's on } B).$$

(5) Let $(\eta_t)_{t\geq 0}$ denote the *Ferromagnetic Stochastic Ising Model (FSIM)* on $\mathbb{Z}^d$ at given inverse temperature $\beta \in (0,\infty)$, i.e., the *spin-flip* Markov process on $\Omega := \{-1,1\}^{\mathbb{Z}^d}$ with local tranisition rates:

$$c(x,\eta) = \exp -\beta\eta(x) \sum_{y:\|x-y\|=1} \eta(y), \quad x \in \mathbb{Z}^d, \eta \in \Omega$$

(a) Show that the FSIM is an *attractive* system and explain what this property means.

(b) Denote by $\subseteq$ the partial order on Ω defined as follows: given $\eta, \eta' \in \Omega$, $\eta \subseteq \eta'$ iff $\eta(x) \leq \eta'(x)$ for all $x \in \mathbb{Z}^d$. Consider two instances of the FSIM, $(\eta^(1)_t)_{t\geq 0}$ and $(\eta^(1)_t)_{t\geq 0}$, starting from $\eta^(1)_0$ and $\eta^(2)_0$, resepctively, such that $\eta^(1)_0 \subseteq \eta^(2)_0$. Prove that there exists a coupling of these two instances such that, with probability 1, $\eta^(1)_t \subseteq \eta^(2)_t$ for all $t > 0$.

NEW exercises -Give the definition of bond percolation on a given connected graph. - Assume that the underlying graph is $\mathbb{Z}^d$ and define the so-called cluster at the origin. -show that BP can be constructed in such way that all clusters are coupled. -Define Invasion percolation -Sketch the main line

(a) Let μ be a probability measure on $\mathbb{R}$. Given two real–valued non–decreasing functions f and g, show that μ satisfies the so–called FKG inequality, that is:

$$\mu[fg] \geq \mu[f]\mu[g], \tag{3}$$

where $\mu[f]$ denotes the expectation of f with respect to μ.

(b) Consider now a finite set S and let μ be a probability measure on the power set $\mathcal{P}(S)$ associated to S. Is it possible to state the FKG inequality as in equation (3) in this setting? Explain what a *monotone* real–valued function $f : \mathcal{P}(S) \longrightarrow \mathbb{R}$ means in this setting, and which assumption on μ suffices.

(6) (a) Define the so–called *directed site percolation* on $\mathbb{Z}^2$, the associated random subset called *cluster of the origin*, and the so–called *critical percolation threshold* denoted by p_c.

(b) Sketch the main lines of the proof of the following upper bound :

$$p_c \leq 80/81.$$

(7) (a) Give the definition of the *contact process* with intensity $\lambda > 0$ on $\{0,1\}^{\mathbb{Z}^d}$.

(b) Denote by P^λ the semigroup of the *contact process* with intensity λ, and by $\delta_{[1]}$ the delta measure associated to the configuration $[1]$ with all 1's. Show that, for all $t \geq 0$, the following stochastic domination holds:

$$\delta_{[1]} P_t^{\lambda_1} \preceq \delta_{[1]} P_t^{\lambda_2},$$

with $0 < \lambda_1 < \lambda_2 < \infty$.