Re-examination for the course on

Random Walks

Teacher: L. Avena

Friday 03 February 2017, 10:00–13:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a **full** explanation.
- The use of notes or diktaat is **not allowed**.
- There are 18 questions. The total number of points is 100 + *bonus* (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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(1) **[5]** Consider a symmetric random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ starting from the origin with the following jumping rules: from any $x \in \mathbb{Z}$, it jumps either to $x+2$ or to $x-2$ with probability $1/2$. Compute the probability $\mathbb{P}_0(S_M = x)$ for arbitrary $M \in \mathbb{N}_0$ and $x \in \mathbb{Z}$.

(2) Consider the simple random walk $(S_n)_{n\in\mathbb{N}_0}$ on the d-dimensional integer lattice $\mathbb{Z}^d$.

(a) **[5]** Give the definitions of *recurrence* and *transience*.

(b) **[10]** For which d is the simple random walk transient? Explain the main lines of a possible proof?

(3) Consider an infinite rooted regular tree with degree $d \geq 1$ (i.e. each node has d children) and unit resistances on the edges.

(a) **[5]** Compute the effective resistance as a function of d.

(b) **[5]** For which d's is the effective resistance finite? What does this imply for the random walk associated to this infinite d-regular tree?

(4) Consider a finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with conductances $C_{xy} \in (0,\infty)$ assigned to each of the edges $xy \in \mathcal{E}$ such that $C_{xy} = C_{yx}$. Pick any two vertices $a, b \in \mathcal{V}$ with $a \neq b$ and place a battery across them.

(a) **[10]** After defining the necessary objects, state the related Dirichlet, Thomson and Rayleigh monotonicity principles.

(b) **[10]** Give an application of the Rayleigh monotonicity principle.

(5) Let c_n denote the number of self-avoiding walks of length $n \in \mathbb{N}$ on the ladder (i.e., two parallel copies of $\mathbb{Z}$ that are sideways connected).

(a) **[5]** Compute c_4.

(b) **[5]** Show that $3 \times 2^n \leq c_{3n+1} \leq 3 \times 2^{3n}$, $n \in \mathbb{N}$, and use this to obtain bounds on the connectivity constant μ.

(6) (a) [5] Define the path space $\mathcal{W}_n$ and the path measure P_n^ζ of the *pinning* polymer model of length $n \in \mathbb{N}$.

(b) [5] Let $\zeta \mapsto f(\zeta)$ be the *free energy* associated with the pinned polymer, we saw that

$$f(\zeta) = \begin{cases} 0, & \text{if } \zeta \leq 0, \\ \frac{1}{2}\zeta - \frac{1}{2}\log(2 - e^{-\zeta}), & \text{if } \zeta > 0. \end{cases} \tag{1}$$

Compute and draw a plot of the derivative of $f(\zeta)$ and explain how to read from this plot the phase transition *localized versus delocalized.*

(c) [5] Define the same objects as in question 6.a for the *wetting* polymer model of length $n \in \mathbb{N}$.

(d) **[Bonus]** Show how to obtain the free energy associated with the wetting polymer model by knowing the one associated with the pinned polymer.

(7) (a) [5] Explain how the simple random walk on $\mathbb{Z}$ is related to the Brownian motion on $\mathbb{R}$.

(b) [5] Let $(W(t))_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}$. Set $X(t) = W(2t) - W(t)$. Is $(X(t))_{t\geq 0}$ a Brownian motion?

(c) [5] Let $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ be independent standard Brownian motions on $\mathbb{R}$ and let $a, b, c \in \mathbb{R} \setminus \{0\}$. What is $\left(c\left[aW_1(t/a^2) + bW_2(t/b^2)\right]\right)_{t\geq 0}$ equal to in distribution?

(8) Consider the one period binomial asset pricing model. Suppose the the current price of a stock is $S_0 = 10$ euro, and that at the end of the period its price must be either $S_1 = 5$ or $S_1 = 20$ euro. A call option on the stock is available with a strike price of $K = 10$ euro, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest.

(a) [5] Compute the arbitrage-free price of the call option.

(b) [5] Suppose that you can buy such an option on the market for 3 euro. What should you do?