Re–Examination for the course:

**Random Walks**

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Wednesday 10 February 2016, 14:00–17:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or diktaat is not allowed.
- There are 10 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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(1) [5] Given three stopping times $T_1$, $T_2$ and $T_3$, let $T = \min\{T_1, T_2, T_3\}$ be their minimum. Is $T$ again a stopping time? Prove your answer!

(2) Consider simple random walk $(S_n)_{n \in \mathbb{N}_0}$ on the integer lattice $\mathbb{Z}$.

   (a) [5] State the related Law of large numbers, Central limit theorem and Large Deviation Principle.

   (b) [5] Describe the meaning of the three theorems in point (2.a) above and comment on the relations among them.

(3) [10] Consider a random walk on the square lattice $\mathbb{Z}^2$ with “diagonal jumps of size 2”, i.e., the jump probabilities are

   \[
P(X_1 = x) = \begin{cases} 
   \frac{1}{4}, & \text{if } x \in \{(2,2), (-2,2), (2,-2), (-2,-2)\}, \\
   0, & \text{otherwise.}
   \end{cases}
\]

Compute the covariance matrix $(\text{Cov}(X_1^{(i)}, X_1^{(j)}))_{i,j=1,2}$, where $X_1^{(i)}$ denotes the $i$-th component of $X_1$. State the central limit theorem for the partial sums $S_n = \sum_{i=1}^{n} X_i$, $n \in \mathbb{N}$.

(4) [5] Compute the effective resistance between $a$ and $b$ of the following network of unit resistances:

![Network Diagram]
(5) Consider the finite piece of the integer lattice \( \mathbb{Z} \) containing the first \( N + 1 \) non-negative vertices \( \{0, 1, \ldots, N - 1, N\} =: \mathcal{V} \). Let \( f : \mathcal{V} \to \mathbb{R} \) be an arbitrary function satisfying the following Dirichlet problem:

\[
\begin{align*}
    f(x) &= \frac{1}{2} f(x + 1) + \frac{1}{2} f(x - 1) \quad x \in \mathcal{V} \setminus \{0, N\}, \\
    f(0) &= 0, \\
    f(N) &= 1.
\end{align*}
\]

(a) [5] For \( x \in \mathcal{V} \), let \( p(x) \) be the probability that the simple random walk on this finite piece of \( \mathbb{Z} \) starting from \( x \) hits \( N \) before \( 0 \), that is, \( p(x) = P_x(\tau_N < \tau_0) \) where \( \tau_N \) and \( \tau_0 \) denote the hitting times of \( N \) and \( 0 \), respectively. Prove that \( p(x) \) satisfies the Dirichlet principle above.

(b) [10] Prove that the Dirichlet problem above admits a unique solution. Deduce that the probability \( p(x) \) in point (4.a) can be interpreted in terms of a voltage.

(6) Let \( c_n \) denote the number of self-avoiding walks of length \( n \in \mathbb{N} \) on the ladder (i.e., two parallel copies of \( \mathbb{Z} \) that are sideways connected).

(a) [5] What inequality is satisfied by \( n \mapsto c_n \), and why does this inequality imply the existence of the so-called connective constant \( \mu \)?

(b) [5] Compute \( c_4 \).

(c) [5] Show that \( 3 \times 2^n \leq c_{3n+1} \leq 3 \times 2^{3n}, n \in \mathbb{N} \), and use this to obtain bounds on \( \mu \).

(b) [10] Consider the function \( \zeta \mapsto f_n(\zeta) = \frac{1}{n} \log Z_n^\zeta \). Compute and show how its first and second derivatives are related to the fraction of absorbed monomers (i.e., points of the path on the horizontal line).

(c) [5] Denote by \( E_{n,rw} \) the expectation with respect to the simple random walk of size \( n \) and let \( Z_n^{\zeta, 0} = E_{n,rw}^\zeta [e^{\zeta \sum_{i=1}^{n} 1(S_i = 0)} 1_{\{S_n = 0\}}] \) be the partition function of the pinned polymer of size \( n \) constrained to end at the membrane (i.e. on the horizontal line). Set \( f_n^0(\zeta) := \frac{1}{n} \log Z_n^{\zeta, 0} \) and show that for every \( \zeta \), the sequence \( \{f_n^0(\zeta)\}_{n \geq 0} \) admits limit \( f_0(\zeta) = \lim_{n \to \infty} f_n^0(\zeta) \). Consequently, use the fact that

\[
Z_n^{\zeta, 0} \leq Z_n^\zeta \leq (1 + Cn)Z_n^{\zeta, 0}
\]

for some constant \( C \in (0, \infty) \), to show that \( f_0(\zeta) = f(\zeta) \), with \( f(\zeta) \) being the free energy of the original polymer model without the constraint to end at the membrane.

(d) [Bonus] Prove that \( Z_n^{\zeta, 0} \leq Z_n^\zeta \leq (1 + Cn)Z_n^{\zeta, 0} \). (HINT: you may want to use that \( a(k)/b(k) \leq Ck \) for all \( k \in 2\mathbb{N} \) and some \( C \in (0, \infty) \) where \( a(k) = P(\sigma > k) \), \( b(k) = P(\sigma = k) \), and \( \sigma \) stands for the first return time to 0 of the simple random walk.)
(7) [10] Give a definition of the one-dimensional Wiener process (also known as standard Brownian motion) \((W_t)_{t\geq 0}\) and say as much as possible on its relation with the simple random walk.

(8) Suppose that the current price of a stock is \(S_0 = 100\) euro, and that at the end of a period of time its price must be either \(S_1 = 75\) or \(S_1 = 150\) euro. A call option on the stock is available with a striking price of \(K = 90\) euro, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest.

(a) [5] Compute the arbitrage-free price of the call option.

(b) [5] Suppose that you can buy such an option on the market for \(€20\). What should you do? Explain your answer.