Re–Examination for the course:

Random Walks

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Wednesday 10 February 2016, 14:00–17:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or diktaat is not allowed.
- There are 10 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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(1) **[5]** Given three stopping times T_1, T_2 and T_3, let $T = \min\{T_1, T_2, T_3\}$ be their minimum. Is T again a stopping time? Prove your answer!

(2) Consider simple random walk $(S_n)_{n\in\mathbb{N}_0}$ on the integer lattice $\mathbb{Z}$.

(a) **[5]** State the related Law of large numbers, Central limit theorem and Large Deviation Principle.

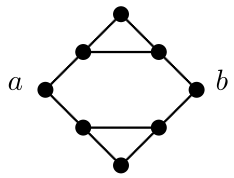
(b) **[5]** Describe the meaning of the three theorems in point (2.a) above and comment on the relations among them.

(3) **[10]** Consider a random walk on the square lattice $\mathbb{Z}^2$ with "diagonal jumps of size 2", i.e., the jump probabilities are

$$P(X_1 = x) = \begin{cases} \frac{1}{4}, & \text{if } x \in \big\{(2,2), (-2,2), (2,-2), (-2,-2)\big\}, \\ 0, & \text{otherwise.} \end{cases}$$

Compute the covariance matrix $(\mathrm{Cov}(X_1^{(i)}, X_1^{(j)}))_{i,j=1,2}$, where $X_1^{(i)}$ denotes the i-th component of X_1. State the central limit theorem for the partial sums $S_n = \sum_{i=1}^{n} X_i$, $n \in \mathbb{N}$.

(4) **[5]** Compute the effective resistance between a and b of the following network of unit resistances:



(5) Consider the finite piece of the integer lattice $\mathbb{Z}$ containing the first $N+1$ non-negative vertices $\{0, 1, \ldots, N-1, N\} =: \mathcal{V}$. Let $f : \mathcal{V} \to \mathbb{R}$ be an arbitrary function satisfying the following Dirichlet problem:

$$\begin{cases} f(x) = \frac{1}{2}f(x+1) + \frac{1}{2}f(x-1) & x \in \mathcal{V} \setminus \{0, N\}, \\ f(0) = 0, \\ f(N) = 1. \end{cases}$$

(a) **[5]** For $x \in \mathcal{V}$, let $p(x)$ be the probability that the simple random walk on this finite piece of $\mathbb{Z}$ starting from x hits N before 0, that is, $p(x) = P_x(\tau_N < \tau_0)$ where τ_N and τ_0 denote the hitting times of N and 0, respectively. Prove that $p(x)$ satisfies the Dirichlet principle above.

(b) **[10]** Prove that the Dirichlet problem above admits a unique solution. Deduce that the probability $p(x)$ in point (4.a) can be interpreted in terms of a *voltage*.

(6) Let c_n denote the number of self-avoiding walks of length $n \in \mathbb{N}$ on the ladder (i.e., two parallel copies of $\mathbb{Z}$ that are sideways connected).

(a) **[5]** What inequality is satisfied by $n \mapsto c_n$, and why does this inequality imply the existence of the so-called connective constant μ?

(b) **[5]** Compute c_4.

(c) **[5]** Show that $3 \times 2^n \le c_{3n+1} \le 3 \times 2^{3n}$, $n \in \mathbb{N}$, and use this to obtain bounds on μ.

(6) (a) **[5]** Give the definition of the path space $\mathcal{W}_n$ of the pinned polymer of length $n \in \mathbb{N}$. The path measure with interaction strength $\zeta \in \mathbb{R}$ is

$$\bar{P}_n^{\zeta}(w) = \frac{1}{Z_n^{\zeta}} e^{\zeta \sum_{i=1}^{n} 1_{\{w_i=0\}}} \bar{P}_n(w), \qquad w \in \mathcal{W}_n.$$

Explain what this path measure models.

(b) **[10]** Consider the function $\zeta \mapsto f_n(\zeta) = \frac{1}{n} \log Z_n^{\zeta}$. Compute and show how its first and second derivatives are related to the fraction of *absorbed monomers* (i.e., points of the path on the horizontal line).

(c) **[5]** Denote by E_n^{srw} the expectation with respect to the simple random walk of size n and let $Z_n^{\zeta,0} = E_n^{srw}\left[e^{\zeta \sum_{i=1}^{n} 1_{\{S_i=0\}}} 1_{\{S_n=0\}}\right]$ be the partition function of the pinned polymer of size n *constrained to end at the membrane* (i.e. on the horizontal line). Set $f_n^0(\zeta) := \frac{1}{n} \log Z_n^{\zeta,0}$ and show that for every ζ, the sequence $\{f_n^0(\zeta)\}_{n\ge0}$ admits limit $f^0(\zeta) = \lim_{n\to\infty} f_n^0(\zeta)$. Consequently, use the fact that

$$Z_n^{\zeta,0} \le Z_n^{\zeta} \le (1 + Cn) Z_n^{\zeta,0}$$

for some constant $C \in (0, \infty)$, to show that $f^0(\zeta) = f(\zeta)$, with $f(\zeta)$ being the *free energy* of the original polymer model *without the constraint* to end at the membrane.

(d) **[Bonus]** Prove that $Z_n^{\zeta,0} \le Z_n^{\zeta} \le (1 + Cn) Z_n^{\zeta,0}$. (*HINT*: you may want to use that $a(k)/b(k) \le Ck$ for all $k \in 2\mathbb{N}$ and some $C \in (0, \infty)$ where $a(k) = P(\sigma > k)$, $b(k) = P(\sigma = k)$, and σ stands for the first return time to 0 of the simple random walk.)

(7) **[10]** Give a definition of the one-dimensional Wiener process (also known as standard Brownian motion) $(W_t)_{t \geq 0}$ and say as much as possible on its relation with the simple random walk.

(8) Suppose that the current price of a stock is $S_0 = 100$ euro, and that at the end of a period of time its price must be either $S_1 = 75$ or $S_1 = 150$ euro. A call option on the stock is available with a striking price of $K = 90$ euro, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest.

(a) **[5]** Compute the arbitrage-free price of the call option.

(b) **[5]** Suppose that you can buy such an option on the market for €20. What should you do? Explain your answer.