Retake: Modular Forms	**28 June 2016**
Mastermath	**10:00–13:00**

Note:

- The use of books, lecture notes, calculators, etc. is *not* allowed.
- If you are unable to answer a subitem, you are still allowed to use the result in the remainder of the exercise.
- Throughout this exam, N and k denote positive integers.

1. This question tests your general knowledge of the material. If you use terminology or notation from the course in answering a subquestion, you should define these.

   (a) Let f be a meromorphic function on $\mathbb{H}$ that is weakly modular of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Define what it means for f to be *meromorphic* at the cusp ∞ of $\mathrm{SL}_2(\mathbb{Z})$.

   (b) Let Γ be a congruence subgroup, and let f be a meromorphic function on the upper half-plane $\mathbb{H}$. Define what it means for f to be *weakly modular* of weight k for the group Γ.

   (c) Let Γ be a congruence subgroup and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Show that $\gamma\Gamma\gamma^{-1}$ is a congruence subgroup.

   (d) Let Γ be a congruence subgroup. Assume that Γ does not contain the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. What does it mean for a cusp of Γ to be regular or irregular?

   (e) Let E be an elliptic curve over $\mathbb{Q}$ and let $L(E,s)$ be its L-function. Assume that $L(E,1) \neq 0$. Explain whether $E(\mathbb{Q})$ (the set of $\mathbb{Q}$-rational points on E) is necessarily finite or not.
   (You are allowed to use results that were discussed but not proved during the course.)

2. For even $k \geq 4$, denote by G_k the Eisenstein series of weight k for $\mathrm{SL}_2(\mathbb{Z})$.

   (a) Show that $\mathrm{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ is spanned by all $G_4^a G_6^b$ with $a, b \in \mathbb{Z}_{\geq 0}$ and $4a + 6b = k$.

   (b) Show that G_4 and G_6 are algebraically independent over $\mathbb{C}$.

3. Let $\Gamma := \Gamma_1(2)$. In this exercise you may use that $\dim(\mathcal{E}_{12}(\Gamma)) = \#\mathrm{Cusps}(\Gamma)$, where $\mathcal{E}_{12}(\Gamma)$ denotes the Eisenstein subspace of $M_{12}(\Gamma)$.

   (a) Show that Γ has exactly two cusps and determine representatives of these in $\mathbb{P}^1(\mathbb{Q})$.

   (b) Show that the index $(\mathrm{SL}_2(\mathbb{Z}) : \Gamma)$ equals 3.

   (c) Use the valence formula to show that $\dim(\mathrm{M}_{12}(\Gamma)) \leq 4$.

   (d) Show that $\dim(\mathrm{S}_{12}(\Gamma)) = 2$.

4. Suppose that $\mathrm{S}_k(\Gamma_0(N))$ contains some normalised eigenform g. Write $f = g^2 \in \mathrm{S}_{2k}(\Gamma_0(N))$. Calculate the first two terms of the q-expansions (i.e. the terms involving q and q^2) of f and $T_2 f$, and deduce that the dimension of $\mathrm{S}_{2k}(\Gamma_0(N))$ is at least 2.

Continue on the back

5. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

   (a) Define the *Petersson inner product* $\langle f, g\rangle_\Gamma$ of two modular forms $f, g \in \mathrm{M}_k(\Gamma)$, at least one of which is a cusp form. Give an explicit example to show what goes wrong when one tries to define $\langle f, f\rangle_\Gamma$ in this way if f is not a cusp form.
   (You do not need to show that the definition of $\langle f, g\rangle_\Gamma$ is independent of choices made in the definition.)

   (b) Let m be a positive integer coprime to N, and let $T_m^\dagger$ be the adjoint of the Hecke operator T_m on $\mathrm{S}_k(\Gamma_1(N))$ with respect to the Petersson inner product. Give a formula expressing $T_m^\dagger$ in terms of T_m and the diamond operator $\langle m\rangle$.

   (c) Let f be a Hecke eigenform in $\mathrm{S}_k(\Gamma_0(N))$ (or equivalently a Hecke eigenform in $\mathrm{S}_k(\Gamma_1(N))$ with trivial character), with q-expansion $f = \sum_{m=1}^\infty a_m q^m$, normalised such that $a_1 = 1$. Show that all coefficients a_m, where m is a positive integer coprime to N, are real.

Maximum scores per subitem				
1a: 5	2a: 7	3a: 5	4: 9	5a: 7
1b: 5	2b: 6	3b: 4		5b: 5
1c: 4		3c: 7		5c: 7
1d: 5		3d: 6		
1e: 8				
Maximum total = 90				
Mark = 1 + Total/10				