Algebraic Geometry 1–Retake Exam 24 January 2017

- Time allowed: 3 hours.
- You should answer all 4 questions.
- You may quote results from the lectures without proof in the exam. If you wish to use results from the exercises then you are expected to re-prove them in the exam.
- Pen and paper only allowed - no books, notes, calculators etc.
- Throughout, k denotes an algebraically closed field, and all varieties we consider are varieties over the field k.

1. (a) Give the definition of a variety. Your answer should include the definition of a k-space, and of an affine variety.

   (b) Give the definition of a morphism of varieties. Your answer should include the definition of a morphism of k-spaces.

2. A hyperplane in $\mathbb{P}^n$ is the zero locus of a non-zero linear homogeneous polynomial in $k[x_0, \ldots, x_n]$. Let $H \subset \mathbb{P}^n$ be a hyperplane, and let $Z \subset \mathbb{P}^n$ be an irreducible projective variety of dimension > 0.

   (a) Show that $\mathbb{P}^n \setminus H$ is affine.

   (b) Show that $H \cap Z$ is not empty.

   (c) Show that an irreducible projective curve can be written as the union of two affine open subsets.

3. Assume $\operatorname{char} k \neq 2$. Let $X \subset \mathbb{A}^2$ be the curve defined by the polynomial $y^2 - x^2(x+1)$. Let P be the point $(0,0)$.

   (a) Show that X is irreducible.

   (b) Show that X is not smooth at P.

   (c) Show that $X \setminus P$ is affine.

   (d) Exhibit an isomorphism $\varphi\colon \mathbb{A}^1 \setminus \{-1, +1\} \xrightarrow{\sim} X \setminus \{P\}$. [Hint: try writing $x = t^2 - 1$.]

   (e) Compute $v_P(x)$.

   (f) Find an element g of the function field $K(X) = K(X \setminus \{P\}) \cong K(\mathbb{A}^1 \setminus \{-1, +1\}) = K(\mathbb{A}^1) \cong k(t)$ with $v_P(g) = 1$.

4. (a) Give two affine varieties X and Y whose underlying topological spaces are homeomorphic, but which are not isomorphic.

   (b) Give two varieties X and Y in $\mathbb{P}^1$ which are isomorphic, but not projectively equivalent (i.e. there is no $g \in \mathrm{PGL}_2(k)$ such that g restricts to an isomorphism from X to Y).

   (c) State the definition of a divisor on a smooth projective irreducible surface.

   (d) Let C be a divisor on $\mathbb{P}^1 \times \mathbb{P}^1$. Write Δ for the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, and H, V for the fibres over $(0:1)$ of the two projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Show that

   $$C \cdot \Delta = C \cdot H + C \cdot V,$$

   where the '$\cdot$' means the intersection product.