Tentamen Manifolds 1, 19-1-2016
Exercise 1

a. For any $p \in S^1 \subset \mathbb{R}^2$ write down a parameterization of S^1 around p and derive an explicit expression for T_pS^1.

b. Give an example of a vector field on S^1 that is non-zero at every point.

c. Define $\omega = dx \wedge dy$ on $\mathbb{R}^2$ and X any smooth vector field on S^1. For any $p \in S^1$ and $v \in T_pS^1$ define $\eta(p)(v) = \omega(p)(X(p), v)$. Prove that η is a smooth 1-form on S^1.

d. Compute $d\eta$ for the 1-form from part c.

e. Let $f : S^1 \to \mathbb{R}$ be the map defined by $f(x, y) = xy$. For your vector field X from part b, compute $Df(p)(X(p))$.

Exercise 2 In this exercise we denote the coordinates in $\mathbb{R}^4$ by x^1, x^2, x^3, x^4. Define $\phi, \psi : \mathbb{R}^2 \to \mathbb{R}^4$ by $\phi(s, t) = (s, s + t, t, s - t)$ and $\psi(s, t) = (t, s, s, s)$ and $Im(\phi) = A$ and $Im(\psi) = B$. The plane $\mathbb{R}^2$ is oriented by $ds \wedge dt$.

a. Show that the 2-form $\eta = dx^1 \wedge dx^2$, when restricted to A defines an orientation on A and the same for B.

b. Define a map $F : A \to B$ by $F(p, q, r, s) = (r, s, s, s)$. Is $F : A \to B$ orientation preserving with respect to the orientations of A and B chosen in part a?

c. Compute the curvature of $\pi(A) \subset \mathbb{R}^3$ in the point $(0, 0, 0)$, where $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ is defined by $\pi(x^1, x^2, x^3, x^4) = (x^2, x^3, x^4)$.

Exercise 3 Suppose A and B are oriented compact n-manifolds without boundary, B is connected and $f : A \to B$ is a smooth map.

a. From now on suppose A is the disjoint union of two connected subsets A_1, A_2. Explain why A_1 and A_2 must also be compact smooth oriented n-manifolds.

b. Even though A is disconnected, define $deg(f) = \int_A f^*\omega$ where $\int_B \omega = 1$ for some $\omega \in \Lambda^n(B)$. Prove that $deg(f)$ is independent of the choice of ω.

c. Now assume X is an oriented connected compact $n + 1$ manifold with boundary $A = A_1 \cup A_2$ and $F : X \to B$ is a smooth map. Show that $deg(F|_{A_1}) + deg(F|_{A_2}) = 0$.

Exercise 4

a. Consider two vector fields X, Y with non-degenerate zeros on S^{16}. What is $Index(X) - Index(Y)$? Explain your answer.

b. If M is a compact manifold, and $p, q, r \in M$ are distinct points, show that there exists a smooth function $f : M \to \mathbb{R}$ such that $f(p) = 19$, $f(q) = 1$ and $f(r) = 2016$.