Exam Manifolds 1, 24-1-2017

Exercise 1

In this exercise V, W are finite dimensional vector spaces and $X \subset V$ and $Y \subset W$ are C^k-manifolds without boundary for some $k > 2$. The dimension of X is n and the dimension of Y is m.

a. Prove that the Cartesian product $Z = X \times Y \subset V \times W$ is a C^k-manifold too. What is its dimension?

b. Show that $T_z Z = T_x X \times T_y Y$ for $z = (x, y) \in Z$ and $x \in X$, $y \in Y$.

c. If X, Y are oriented manifolds, is Z necessarily orientable too? If no give an example, if yes give a proof.

d. Take $X = Y = S^1$ the unit circle and $V = W = \mathbb{R}^2$. Compute the matrix for $Df(\pi, \pi)$ with respect to the standard bases, where $f : \mathbb{R}^2 \to Z$ is given by $f(a, b) = (\cos a, \sin a, \cos b, \sin b)$.

e. Compute the pull-back $f^*\omega$ where ω is any 3-form on $\mathbb{R}^4$ restricted to Z.

Exercise 2

Consider a C^∞ function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $df(p) \neq 0$ for any $p \in f^{-1}(\{0\})$. The standard basis of $\mathbb{R}^3$ is denoted $\{e_1, e_2, e_3\}$.

a. Use the Rank Theorem to show that $X = f^{-1}(\{0\})$ is a manifold.

b. Prove that the tangent space $T_x X$ is equal to $\ker(Df(x))$ for any $x \in X$.

c. Define $N(p) = \sum_{i=1}^{3} Df(p)(e_i)e_i$ and $\tilde{N} = \frac{N}{|N|}$ and consider the vector field F defined by $F(p) = e_3 - \langle \tilde{N}(p), e_3 \rangle \tilde{N}(p)$. Show F is a differentiable vector field, tangent to X.

d. Write down the system of ordinary differential equations that an integral curve to F has to satisfy.

e. Assume X is compact, connected and has positive Gauss curvature at every point. Prove that there must be at least one point $q \in X$ such that $T_q X = \mathrm{Span}\{e_1, e_2\}$.

Exercise 3

Consider the 1-form $\alpha \in \Omega^1(\mathbb{R}^3)$ defined by $\alpha = x^1 dx^2 + x^2 dx^3 + x^3 dx^1$. Here x^i denotes the basis dual to the standard basis at every point in $\mathbb{R}^3$. Define $H = \{p \in \mathbb{R}^3 | x^3(p) \geq 0\}$ and $S^2 \subset \mathbb{R}^3$ is the unit sphere.

a. Show that $\omega = dx^1 \wedge dx^2 + dx^2 \wedge dx^3 + dx^3 \wedge dx^1$ satisfies $\omega = d\alpha$.

b. Compute $\int_{S^2 \cap \partial H} \alpha$.

c. Prove that your answer in part b is equal to $\int_{S^2 \cap H} \omega$.

d. What is $\int_{S^2} \omega$?

e. Compute the degree of the map $A : S^2 \to S^2$ given by $A(p) = -p$.