Mathematical Institute	Exam Time Series
Leiden University	June 11, 2014

Give concise, but rigorous arguments. Always motivate your answers. Use of calculators is neither allowed nor useful. Good luck!

1. (a) Give the definition of an m-dependent time series.

   (b) Formulate and prove the central limit theorem for the sample mean $\overline{X}_n = n^{-1}\sum_{t=1}^n X_t$, with X_t an m-dependent time series.

2. (a) Let $A_1, \dots, A_n, B_1, \dots, B_n$ be uncorrelated random variables with

$$\mathbb{E}[A_k] = \mathbb{E}[B_k] = 0, \quad \mathbb{E}[A_k^2] = \mathbb{E}[B_k^2] = \sigma_k^2, \quad k = 1, \dots, n.$$

   Let $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \pi$. Consider the time series X_t defined by

$$X_t = \sum_{k=1}^n A_k \cos(t\lambda_k) + \sum_{k=1}^n B_k \sin(t\lambda_k).$$

   Show that X_t is stationary and determine its spectral measure F. *Hint*: trigonometric identities

$$\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2},$$
$$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$$

   are useful.

   (b) Give the definition of the transfer function $\psi : (-\pi, \pi] \to \mathbb{C}$ of a filter with filter coefficients ψ_j. Compute the transfer function of the filter $Y_t = X_t - X_{t-7}$. Interpret the result.

   (c) Let X_t be a stationary process and let $Y_t = \sum_j \psi_j X_{t-j}$. Give the definition of the gain of a filter. State and prove the theorem relating the spectral measure F_Y of Y_t to the spectral measure F_X of X_t and the gain.

   (d) Assuming the stationary solution to the AR(1) equation $X_t = \varphi X_{t-1} + Z_t$ with white noise Z_t (with variance σ^2) exists, find the spectral density f_X.

   (e) For the stationary density f_X from d) prove the formula $\int_{-\pi}^{\pi} \log f(\lambda) d\lambda = 2\pi\log(\sigma^2/(2\pi))$. This is a particular case of the Kolmogorov-Szegö formula.

Continued on the other side!

3. (a) Give the definition of the ARMA(p, q) process. What does causality of an ARMA process mean?
   (b) Investigate existence of a stationary solution to the following equation,
   $$X_t = 2X_{t-1} + 3X_{t-2} + Z_t,$$
   where Z_t is an i.i.d. Gaussian white noise. If the stationary solution exists, is it also causal?
   (c) Does there exist a solution with $X_0 = 5$ to the equation from b)? If yes, is it stationary?
4. (a) Let X_t be a GARCH$(1, 1)$ process. What is the best predictor of X_t based on the infinite past $X_{t-1}, X_{t-2}, \ldots$ (assume whatever conditions you think are necessary)? What is the best predictor based on the finite past $X_{t-1}, \ldots, X_{t-n}$?
5. Let X_t be a stationary time series.
   (a) Give the definition of a periodogram $I_n(\lambda)$ based on observations $X_1, \ldots, X_n$.
   (b) Prove that the periodogram is an asymptotically unbiased estimator of the spectral density (under some mild assumption).
6. Consider the stationary, causal solution to the AR(1) equation $X_t = \varphi X_{t-1} + Z_t$ with independent Gaussian white noise Z_t with variance σ^2. Suppose you observe $X_1, \ldots, X_n$.
   (a) Find the Yule-Walker estimator of the parameter φ (*Hint*: remember prediction equations). Next use this to find an estimator of σ^2.
   (b) Compute the maximum likelihood estimator (conditional, or approximate) for the parameter pair (φ, σ^2).

Norming

1(a): 1; **1(b)**: 3.
2(a): 2; **2(b)**: 1; **2(c)**: 2; **2(d)**: 1; **2(e)**: 1.
3(a): 1; **3(b)**: 2; **3(c)**: 1.
4(a): 1.
5(a): 1; **5(b)**: 2.
6(a): 2; **6(b)**: 2.

The final grade depends on the results of the homework assignments as well.