Examination for the course on

Random Walks

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- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or diktaat is not allowed.
- There are 10 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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(1) **[5]** Consider simple random walk on $\mathbb{Z}$. Given two stopping times T_1 and T_2, is the minimum $T = \min\{T_1, T_2\}$ again a stopping time? Prove your answer!

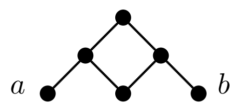
(2) **[10]** Consider a random walk on the square lattice $\mathbb{Z}^2$ with “diagonal jumps”, i.e., the jump probabilities are

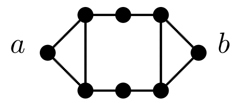
$$P(X_1 = x) = \begin{cases} \frac{1}{4}, & \text{if } x \in \big\{(1,1), (-1,1), (1,-1), (-1,-1)\big\}, \\ 0, & \text{otherwise.} \end{cases}$$

Compute the covariance matrix $(\mathrm{Cov}(X_1^{(i)}, X_1^{(j)}))_{i,j=1,2}$, where $X_1^{(i)}$ denotes the i-th component of X_1. State the central limit theorem for the partial sums $S_n = \sum_{i=1}^n X_i$, $n \in \mathbb{N}$.

(3) **[10]** In the game *double or loose* you bet 1 euro per round. The bet is either doubled or lost, both with 50% chance. The strategy of a gambler is to continue playing until either a total of 10 euro is won (the gambler leaves the game happy) or four times in a row a loss is suffered (the gambler leaves frustrated). Is the expected payoff of this strategy positive, zero or negative? Prove your answer!

(4) Compute the effective resistance between a and b of the following two networks of unit resistances:

(a) **[5]** 

(b) **[5]** 

(5) Given is a finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and two vertices $a, b \in \mathcal{V}$.

(a) [**5**] What is a unit potential from a to b?

(b) [**5**] What is a unit flow from a to b?

(6) Let c_n denote the number of self-avoiding walks of length $n \in \mathbb{N}$ on the triangular lattice (i.e., the two-dimensional lattice where unit triangles are packed together).

(a) [**5**] What inequality is satisfied by $n \mapsto c_n$, and why does this inequality imply the existence of the so-called connective constant μ?

(b) [**5**] Compute c_3.

(c) [**5**] Show that $2^n \leq c_n \leq 6 \times 5^{n-1}$, $n \in \mathbb{N}$ and use this to obtain bounds on μ.

(7) (a) [**5**] Give a description of the path space $\mathcal{W}_n$ of the pinned polymer of length $n \in \mathbb{N}$. The path measure with interaction strength $\zeta \in \mathbb{R}$ is

$$\bar{P}_n^{\zeta}(w) = \frac{1}{Z_n^{\zeta}}\, e^{\zeta \sum_{i=1}^{n} 1_{\{w_i=0\}}}\, \bar{P}_n(w), \qquad w \in \mathcal{W}_n.$$

Explain what this path measure models.

(b) [**5**] Give the definition of the free energy $\zeta \mapsto f(\zeta)$, and explain why this quantity is capable of detecting a phase transition.

(c) [**5**] Give the formula that expresses $f(\zeta)$ in terms of the generating function for the probability distribution of the first return time to the origin of one-dimensional simple random walk.

(d) [**Bonus**] Explain how this formula is derived.

(8) Let $(W_t)_{t \geq 0}$ and $(\tilde{W}_t)_{t \geq 0}$ be independent standard Brownian motions. Put $X_t = \alpha W_t + \beta \tilde{W}_t$, where and $\alpha, \beta \in \mathbb{R}$ are such that $\alpha^2 + \beta^2 = 1$.

(a) [**5**] Show that $(X_t)_{t \geq 0}$ is standard Brownian motion as well.

(b) [**5**] Compute the correlation coefficient $\rho(X_t, W_t)$.

(9) [**5**] Let $(W(t))_{t \geq 0}$ be standard Brownian motion, and let $0 < t_1 < t_1 + t_2 < t_1 + t_2 + t_3$. Compute

$$E\big[W(t_1)W(t_1 + t_2)W(t_1 + t_2 + t_3)\big].$$

(10) Suppose that the current price of a stock is $S_0 = 50$ euro, and that at the end of a period of time its price must be either $S_1 = 25$ or $S_1 = 100$ euro. A call option on the stock is available with a striking price of $K = 50$ euro, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest.

(a) [**5**] Compute the arbitrage-free price of the call option.

(b) [**5**] Suppose that you can buy such an option on the market for €15. What should you do?

SOLUTIONS

(1) By definition, for $n \in \mathbb{N}_0$,

$$\{T_i = n\} \subseteq \mathcal{A}_n \qquad (i = 1, 2).$$

Since $n \mapsto \mathcal{A}_n$ is non-decreasing, we have

$$\{T_i \leq n\} \subseteq \mathcal{A}_n \qquad (i = 1, 2).$$

Hence

$$\{T \leq n\} \subseteq \{T_1 \leq n\} \cup \{T_2 \leq n\} \subseteq \mathcal{A}_n \qquad (i = 1, 2).$$

(2) We have

$$\mathrm{Cov}(X_1^{(i)}, X_1^{(j)}) = \sum_{x \in \mathbb{Z}d} x^{(i)}\, x^{(j)}\, P(X = x) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Also, $E(X_1) = 0$. Consequently, the central limit theorem states that $n^{-1/2} \sum_{k=1}^{n} X_k$ converges in distribution to a $\mathcal{N}(0, \mathrm{Id})$ distribution. (Watch out that there is no scaling factor $1/d$ in front of Id. The individual jumps are longer by a factor $\sqrt{d}$, and this is reflected through the higher variance in the scaling limit.)

(3) This game is a game system in the sense of Section 1.2 of the lecture notes. The object of interest is the minimum of two stopping times, as in Exercise (1). Hence the expected payoff is zero.

(4) (a) The square on the top may be replaced by a single edge of unit resistance. Hence the effective resistance is 3.

(b) By symmetry, the potential on the top is the same as on the bottom. In particular, no current flows vertically. Hence we can remove the two vertical edges. In the resulting graph each of the two parallel arms has resistance 4, and hence the effective resistance is 2.

(5) (a) A unit potential from a to b is a function $u\colon \mathcal{V} \to \mathbb{R}$ such that $u_a = 1$ and $u_b = 0$.

(b) A unit flow from a to b is a function $j\colon \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ such that: (1) $j_{xy} = -j_{yx}$ for all $x, y \in \mathcal{V}$; (2) $\sum_{y \in \mathcal{V}} j_{xy} = 0$ for all $x \in \mathcal{V} \backslash \{a, b\}$; (3) $j_{xy} = 0$ when $xy \notin \mathcal{E}$; (4) $j_a = \sum_{y \in \mathcal{V}} j_{ay} = 1$, $j_b = \sum_{y \in \mathcal{V}} j_{by} = -1$.

(6) (a) The concatenation of two self-avoiding walks of lengths m and n may or may not result in a self-avoiding walk of length $m + n$. The concatenation captures all of them, and so $c_{m+n} \leq c_m c_n$. By Fekete's lemma this implies the existence of $\mu = \lim_{n \to \infty} (c_n)^{1/n}$.

(b) $c_3 = 6 \times 5 \times 5 - 2 \times 6 = 6 \cdot (3 \times 5 + 2 \times 4) = 138$.

(c) Upper bound: first step 6 possibilities, each subsequent step at most 5 possibilities. Lower bound: No self-intersection occurs in a path that only makes steps east of north-east.

(7) (a) The set of n-step directed paths starting at the origin is

$$\mathcal{W}_n = \{w = (i, w_i)_{i=0}^{n}\colon\ w_0 = 0,\ w_{i+1} - w_i \in \{-1, +1\}\ \forall\, 0 \leq i < n\}.$$

For $\zeta \in \mathbb{R}$ the path measure on $\mathcal{W}_n$ for the pinned polymer is

$$\bar{P}_n^\zeta(w) = \frac{1}{Z_n^\zeta} e^{\zeta \sum_{i=1}^n 1_{\{w_i=0\}}} \bar{P}_n(w), \qquad w \in \mathcal{W}_n.$$

where $\bar{P}_n$ is the path measure of directed simple random walk, and Z_n^ζ is the normalising partition function. The exponent counts the number of times the polymer hits the interface. Each hit contributes an energy ζ to the energy, and accordingly is rewarded by a factor e^ζ.

(b) The free energy is defined as $f(\zeta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n^\zeta$. The derivative $f'(\zeta)$ equals the limiting fraction of monomers at the interface.

(c) For $\zeta \le 0$: $f(\zeta) = 0$. For $\zeta > 0$: $f(\zeta) = r(\zeta)$ with $r(\zeta)$ the unique solution of the equation $B(e^{-r}) = e^{-\zeta}$, where $B(z) = F(0;z) = \sum_{n\in\mathbb{N}} z^n P(\sigma_0 = n)$ with σ_0 the first return time to the origin.

(d) See the derivation in the lecture notes Section 5.1, which computes with the partition function for paths that are constrained to end at the interface.

(8) (a) The four characterising properties of standard Brownian motion are satisfied: (i) $X_0 = 0$; (ii) with probability one, $t \mapsto X(t)$ is continuous; (iii) $X(t+s) - X(t)$ is $\mathcal{N}(0,s)$-distributed; (iv) independent increments. Properties (i), (ii) and (iv) are immediate, while property (iii) follows from

$$X(t+s) - X(s) = \alpha[W(t+s) - W(t)] + \beta[\tilde{W}(t+s) - \tilde{W}(t)]$$

and the fact that $\alpha X_1 + \beta X_2 = (\alpha^2 + \beta^2)X_3$ in distribution when X_1, X_2, X_3 are $\mathcal{N}(0,s)$-distributed and X_1, X_2 are independent.

(b) Compute

$$\begin{aligned} E(X_t W_t) &= \alpha E(W_t^2) + \beta E(\tilde{W}_t W_t) = \alpha t, \\ E(X_t^2) &= \alpha^2 E(W_t^2) + \beta^2 E(\tilde{W}_t) = t, \\ E(W_t^2) &= t, \end{aligned}$$

and hence

$$\rho(X_t, W_t) = \frac{E(X_t W_t) - E(X_t)E(W_t)}{\sqrt{E(X_t^2)E(W_t^2)}} = \frac{\alpha t}{t} = \alpha.$$

(9) Compute

$$E = E[W(t_1)W(t_1+t_2)W(t_1+t_2+t_3)] = E[X(X+Y)(X+Y+Z)],$$

where X, Y, Z are independent with mean 0. All powers are odd, and hence $E = 0$.

(10) (a) We have $S_0 = 50$, $S_1(H) = 100 = uS_0$, $S_1(H) = 25 = dS_0$. Hence $u = 2$, $d = \frac{1}{2}$ and

$$q = \frac{1+r-d}{u-d} = \frac{1.25 - 0.5}{2 - 0.5} = \frac{0.75}{1.5} = \frac{1}{2}.$$

Since the pay-off of a call option is $C(\omega) = (S_1(\omega) - 50)^+$, we get

$$C_0 = \frac{1}{1+0.25}\left(\frac{1}{2} \times 50 + \frac{1}{2} \times 0\right) = \frac{50}{2.5} = 20.$$

(b) The number of shares in the replicating portfolio is

$$\Delta_0 = \frac{C_1(H) - C_1(T)}{S_1(H) - S_1(T)} = \frac{50}{75} = \frac{2}{3}.$$

Hence, with the initial wealth of 20, we can replicate the option by buying two thirds of a share for $\frac{2}{3} \times 50 = \frac{100}{3} = 33\frac{1}{3}$ euro, and borrowing $33\frac{1}{3} - 20 = 13\frac{1}{3}$ on the market with interest $r = 25\%$. If the option is sold for 15 euro, then we should proceed as follows:

- short sell $\Delta_0 = \frac{2}{3}$ share of the stock, which gives $\frac{100}{3} = 33\frac{1}{3}$ euro;
- buy one option at 15 euro, and put the difference $33\frac{1}{3} - 15 = 18\frac{1}{3}$ into the money market.

At the end of the period we have to return the share, i.e., pay $\frac{2}{3}S_1(\omega)$, which is $\frac{200}{3} = 66\frac{2}{3}$ euro if $\omega = H$ and $\frac{50}{3} = 16\frac{2}{3}$ euro if $\omega = T$. The money market will give us

$$\Big(\frac{100}{3} - 15\Big)\frac{5}{4} \approx 22.92$$

and the option will give us either 50 or 0 euro. Thus our position will be either $50+22.92 = 72.92$ or $0 + 22.92$, while the corresponding obligations are $\frac{200}{3} \approx 66.67$ or $\frac{50}{3} \approx 16.67$. Hence, our position is covered and in both cases we generate the same profit:

$$\begin{aligned} 50 + \Big(\frac{100}{3} - 15\Big)\frac{5}{4} - \frac{200}{3} &= 50 + \frac{125}{3} - \frac{75}{4} - \frac{200}{3} = \frac{25}{4} = 6.25, \\ \Big(\frac{100}{3} - 15\Big)\frac{5}{4} - \frac{50}{3} &= \frac{125}{3} - \frac{75}{4} - \frac{50}{3} = \frac{25}{4} = 6.25. \end{aligned}$$