Examination for the course on
Random Walks
Teacher: F. den Hollander
Thursday 29 January 2015, 14:00–17:00

• Write your name and student identification number on each piece of paper you hand in.
• All answers must come with a full explanation.
• The use of notes or lecture notes is not allowed.
• There are 10 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

(1) [5] Consider simple random walk \((S_n)_{n \in \mathbb{N}_0}\) on \(\mathbb{Z}\). Compute the Green function \(G(0; z) = \sum_{n \in \mathbb{N}_0} z^n P(S_n = 0), z \in (0, 1)\). \(\text{Hint: } \sum_{m \in \mathbb{N}_0} \binom{2m}{m} u^{2m} = 1/\sqrt{1 - 4u^2}, u \in (0, 1/2)\). 

(2) [5] Consider simple random walk \((S_n)_{n \in \mathbb{N}_0}\) on \(\mathbb{Z}^2\). Give an example of a non-constant random variable \(T\) that is a stopping time, and a \(T\) that is not a stopping time. Prove your answer!

(3) [5] In the game up or down, in each round your capital either increases by 1 euro or decreases by 1 euro, each with probability \(1/2\). The game stops when your capital is 10 euro (you leave happy) or 0 euro (you leave frustrated). You start with 1 euro. Is the expected gain in your capital at the end of the game positive, zero or negative? Prove your answer!

(4) Compute the effective resistance between \(a\) and \(b\) of the following two networks of unit resistances:

(a) [5] \(a\) \(\bullet\) \(\bullet\) \(\bullet\) \(b\)

(b) [10] \(a\) \(\bullet\) \(\bullet\) \(\bullet\) \(\bullet\) \(\bullet\) \(b\)
(5) Given is a finite connected graph \( G = (V, \mathcal{E}) \) and two vertices \( a, b \in V \) with \( a \neq b \).

(a) [5] Use the Dirichlet Principle to write down a formula for the effective resistance \( R_{\text{eff}} \) between \( a \) to \( b \) in terms of unit potentials.

(b) [5] Use the Thomson Principle to write down a formula for \( R_{\text{eff}} \) in terms of unit flows.

Explain the symbols in your answers.

(6) Let \( c_n \) denote the number of self-avoiding walks of length \( n \in \mathbb{N} \) on the ladder (i.e., two parallel copies of \( \mathbb{Z} \) that are sideways connected).

(a) [5] What inequality is satisfied by \( n \mapsto c_n \), and why does this inequality imply the existence of the so-called connective constant \( \mu \)?

(b) [5] Compute \( c_4 \).

(c) [5] Show that \( 3 \cdot 2^n \leq c_{3n+1} \leq 3 \cdot 2^{3n}, n \in \mathbb{N}, \) and use this to obtain bounds on \( \mu \).

(7) (a) [5] Give the formula for the path space \( \mathcal{W}_n^+ \) of the wetted polymer of length \( n \in \mathbb{N} \).

The path measure with interaction strength \( \zeta \in \mathbb{R} \) is

\[
\bar{P}_n^+(w) = \frac{1}{Z_n^\zeta} e^{\zeta \sum_{i=1}^{n} 1_{\{w_i = 0\}}} \bar{P}_n(w), \quad w \in \mathcal{W}_n^+,
\]

where \( \bar{P}_n^+ \) is the uniform measure on \( \mathcal{W}_n^+ \). What does \( Z_n^\zeta \) stand for? Explain what physical setting this path measure models.

(b) [5] Give the definition of the free energy \( \zeta \mapsto f^+(\zeta) \), and explain why this quantity is capable of detecting a phase transition.

(c) [5] Give the formula that expresses \( f^+(\zeta) \) in terms of \( F(0; z) = \sum_{n \in \mathbb{N}} z^n P(\sigma_0 = n) \), \( z \in (0, 1) \), the generating function for the first return time of simple random walk on \( \mathbb{Z} \).

(d) [Bonus] Explain how this formula is derived.

(8) (a) [5] Explain how \( (W(t))_{t \geq 0} \), standard Brownian motion on \( \mathbb{R} \), arises as the scaling limit of \( (S_n)_{n \in \mathbb{N}_0} \), simple random walk on \( \mathbb{Z} \).

(b) [5] Let \( (W(t))_{t \geq 0} \) and \( (\bar{W}(t))_{t \geq 0} \) be independent standard Brownian motions on \( \mathbb{R} \).

What is \( (aW(t/a^2), a\bar{W}(t/a^2))_{t \geq 0} \) for \( a, \bar{a} \in (0, \infty) \) ?

(9) [10] Explain why no arbitrage for the Binomial Asset Pricing Model requires the parameters \( d, u, r \) to satisfy \( d < 1 + r < u \) (\( d = \text{downward}, \ u = \text{upward}, \ r = \text{interest} \)).

(10) [10] Suppose that the current price of a stock is \( S_0 = 10 \) euro, and that at the end of a single period of time its price is either \( S_1 = 5 \) euro or \( S_1 = 20 \) euro. A European call option on the stock is available with a strike price of \( K = 12 \) euro, expiring at the end of the period. It is also possible to borrow and lend money at a 10% interest rate. Compute the arbitrage-free price of this option with the help of the Binomial Asset Pricing Model.