Examination for the course on

Random Walks

Teacher: F. den Hollander

Thursday 29 January 2015, 14:00–17:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or lecture notes is not allowed.
- There are 10 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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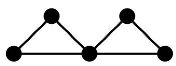
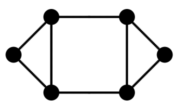
(1) **[5]** Consider simple random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$. Compute the Green function $G(0;z) = \sum_{n\in\mathbb{N}_0} z^n\, P(S_n = 0)$, $z \in (0,1)$. *Hint:* $\sum_{m\in\mathbb{N}_0} \binom{2m}{m} u^{2m} = 1/\sqrt{1-4u^2}$, $u \in (0, \frac{1}{2})$.

(2) **[5]** Consider simple random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}^2$. Give an example of a non-constant random variable T that is a stopping time, and a T that is not a stopping time. Prove your answer!

(3) **[5]** In the game *up or down*, in each round your capital either increases by 1 euro or decreases by 1 euro, each with probability $\frac{1}{2}$. The game stops when your capital is 10 euro (you leave happy) or 0 euro (you leave frustrated). You start with 1 euro. Is the expected gain in your capital at the end of the game positive, zero or negative? Prove your answer!

(4) Compute the effective resistance between a and b of the following two networks of unit resistances:

(a) **[5]** a  b

(b) **[10]** a  b

(5) Given is a finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and two vertices $a, b \in \mathcal{V}$ with $a \neq b$.

(a) **[5]** Use the Dirichlet Principle to write down a formula for the effective resistance R_{eff} between a to b in terms of unit potentials.

(b) **[5]** Use the Thomson Principle to write down a formula for R_{eff} in terms of unit flows.

Explain the symbols in your answers.

(6) Let c_n denote the number of self-avoiding walks of length $n \in \mathbb{N}$ on the ladder (i.e., two parallel copies of $\mathbb{Z}$ that are sideways connected).

(a) **[5]** What inequality is satisfied by $n \mapsto c_n$, and why does this inequality imply the existence of the so-called connective constant μ?

(b) **[5]** Compute c_4.

(c) **[5]** Show that $3 \times 2^n \leq c_{3n+1} \leq 3 \times 2^{3n}$, $n \in \mathbb{N}$, and use this to obtain bounds on μ.

(7) (a) **[5]** Give the formula for the path space $\mathcal{W}_n^+$ of the wetted polymer of length $n \in \mathbb{N}$. The path measure with interaction strength $\zeta \in \mathbb{R}$ is

$$\bar{P}_n^{\zeta,+}(w) = \frac{1}{Z_n^{\zeta,+}} \, e^{\zeta \sum_{i=1}^n 1_{\{w_i=0\}}} \, \bar{P}_n^+(w), \qquad w \in \mathcal{W}_n^+,$$

where $\bar{P}_n^+$ is the uniform measure on $\mathcal{W}_n^+$. What does $Z_n^{\zeta,+}$ stand for? Explain what physical setting this path measure models.

(b) **[5]** Give the definition of the free energy $\zeta \mapsto f^+(\zeta)$, and explain why this quantity is capable of detecting a phase transition.

(c) **[5]** Give the formula that expresses $f^+(\zeta)$ in terms of $F(0; z) = \sum_{n \in \mathbb{N}} z^n P(\sigma_0 = n)$, $z \in (0, 1)$, the generating function for the first return time of simple random walk on $\mathbb{Z}$.

(d) **[Bonus]** Explain how this formula is derived.

(8) (a) **[5]** Explain how $(W(t))_{t \geq 0}$, standard Brownian motion on $\mathbb{R}$, arises as the scaling limit of $(S_n)_{n \in \mathbb{N}_0}$, simple random walk on $\mathbb{Z}$.

(b) **[5]** Let $(W(t))_{t \geq 0}$ and $(\bar{W}(t))_{t \geq 0}$ be independent standard Brownian motions on $\mathbb{R}$. What is $(aW(t/a^2), \bar{a}\bar{W}(t/\bar{a}^2))_{t \geq 0}$ for $a, \bar{a} \in (0, \infty)$?

(9) **[10]** Explain why no arbitrage for the Binomial Asset Pricing Model requires the parameters d, u, r to satisfy $d < 1 + r < u$ (d = downward, u = upward, r = interest).

(10) **[10]** Suppose that the current price of a stock is $S_0 = 10$ euro, and that at the end of a single period of time its price is either $S_1 = 5$ euro or $S_1 = 20$ euro. A European call option on the stock is available with a strike price of $K = 12$ euro, expiring at the end of the period. It is also possible to borrow and lend money at a 10% interest rate. Compute the arbitrage-free price of this option with the help of the Binomial Asset Pricing Model.